

Exercise 6.1 : By using the signal set given in Fig. 6.5 $s_1^a(t) \cdots s_4^a(t)$ and $\psi_1(t)$ and $\psi_2(t)$, repeat the steps of Example 6.2.

Solution : By taking the time waveforms of $s_1^a(t) \cdots s_4^a(t)$ and using $\psi_1(t)$, $\psi_2(t)$ are given in Figs. 6.1 and 6.2

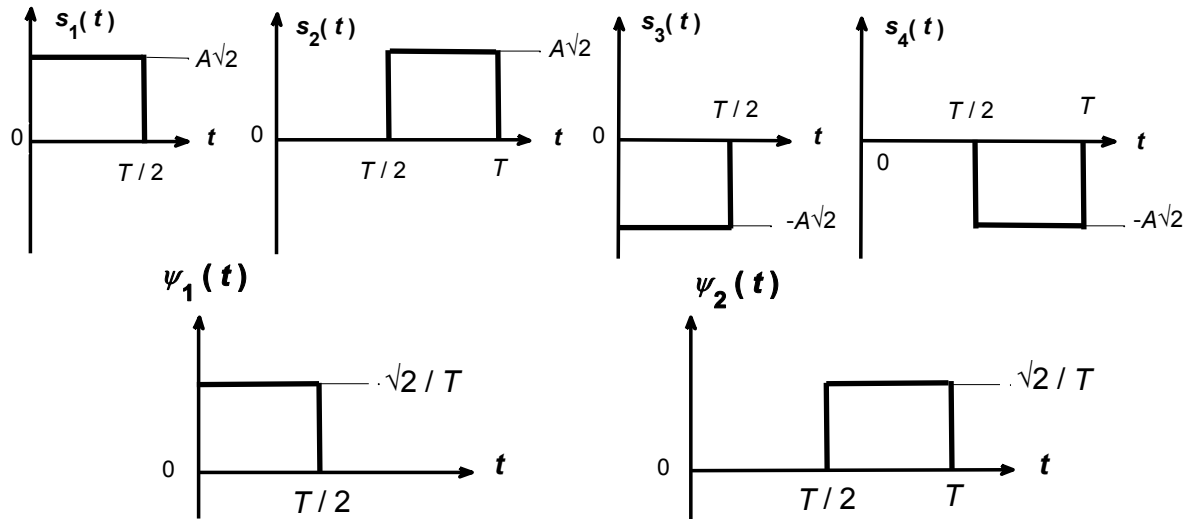


Fig. 6.1 First possible set of signal waveforms, $s_1(t) \cdots s_4(t)$ and orthonormalized basis functions, $\psi_1(t)$, $\psi_2(t)$ for 4 PSK.

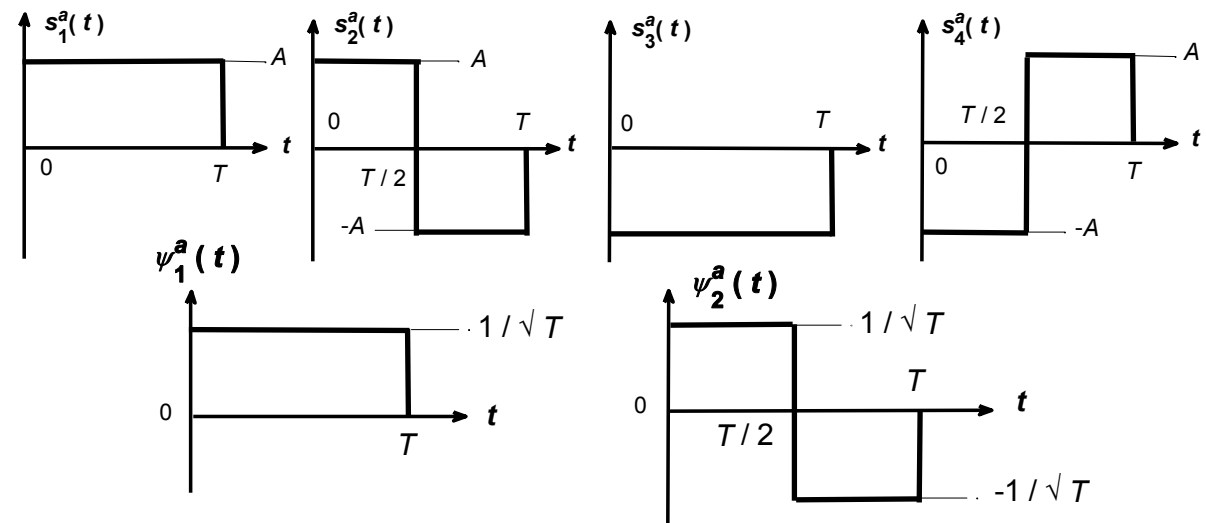


Fig. 6.2 First possible set of signal waveforms, $s_1^a(t) \cdots s_4^a(t)$ and orthonormalized basis functions, $\psi_1^a(t)$, $\psi_2^a(t)$ for 4 PSK.

$$\begin{aligned}
s_1^a(t) &= \begin{cases} A & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} & s_2^a(t) &= \begin{cases} A & 0 \leq t \leq T/2 \\ -A & T/2 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \\
s_3^a(t) &= \begin{cases} -A & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} & s_4^a(t) &= \begin{cases} -A & 0 \leq t \leq T/2 \\ A & T/2 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \\
\psi_1(t) &= \begin{cases} \sqrt{2/T} & 0 \leq t \leq T/2 \\ 0 & \text{otherwise} \end{cases} & \psi_2(t) &= \begin{cases} \sqrt{2/T} & T/2 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{6.1}$$

Now either by eye inspection or by **Gram-Schmidt Orthogonalization Procedure**, we write $s_1^a(t) \cdots s_4^a(t)$ in terms of $\psi_1(t)$ and $\psi_2(t)$. Note that here there is no need to indicate the time intervals, since they are embedded in $\psi_1(t)$ and $\psi_2(t)$

$$\begin{aligned}
s_1^a(t) &= A\sqrt{T/2}\psi_1(t) + A\sqrt{T/2}\psi_2(t) , & s_2^a(t) &= A\sqrt{T/2}\psi_1(t) - A\sqrt{T/2}\psi_2(t) \\
s_3^a(t) &= -A\sqrt{T/2}\psi_1(t) - A\sqrt{T/2}\psi_2(t) , & s_4^a(t) &= -A\sqrt{T/2}\psi_1(t) + A\sqrt{T/2}\psi_2(t) \\
\mathbf{s}_1^a &= [s_{11}, s_{12}] = [A\sqrt{T/2}, A\sqrt{T/2}] , & \mathbf{s}_2^a &= [s_{21}, s_{22}] = [A\sqrt{T/2}, -A\sqrt{T/2}] \\
\mathbf{s}_3^a &= [s_{31}, s_{32}] = [-A\sqrt{T/2}, -A\sqrt{T/2}] , & \mathbf{s}_4^a &= [s_{41}, s_{42}] = [-A\sqrt{T/2}, A\sqrt{T/2}] \\
d_{12} = d_{14} = d_{23} &= A\sqrt{2T} = \sqrt{2\varepsilon_s} , & d_{13} = d_{24} &= 2A\sqrt{T} = 2\sqrt{\varepsilon_s} \\
|\mathbf{s}_1^a| = |\mathbf{s}_2^a| = |\mathbf{s}_3^a| = |\mathbf{s}_4^a| &= A\sqrt{T} , & \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_s &= A^2T
\end{aligned} \tag{6.2}$$

On the second and third lines of (6. 1.2), we have included the vectorial representation of our signals, on the fourth line we have given the respective distance between vector ends, on the last line the length of vectors and the energies are given which can be calculated either from time signals or vectorial representations. As this is PSK, all vector lengths, thus the energies are equal. Now we can plot the constellation diagram of $\mathbf{s}_1^a \cdots \mathbf{s}_4^a$. This is illustrated in Fig. 6.3

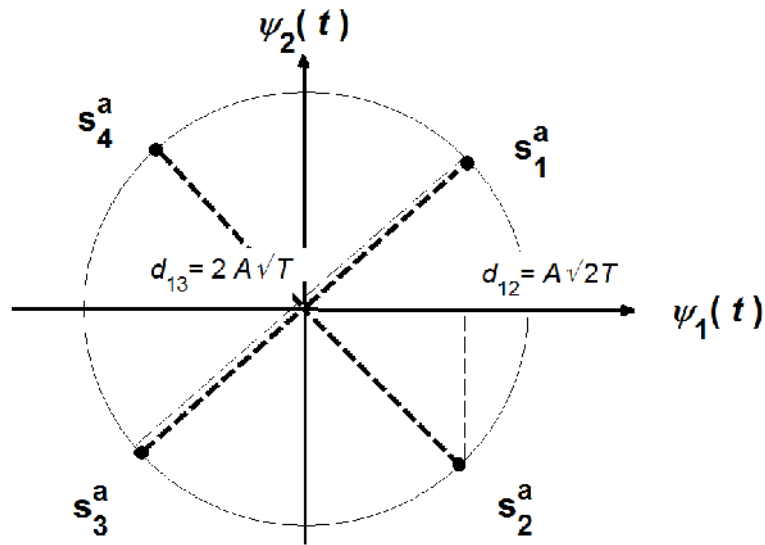
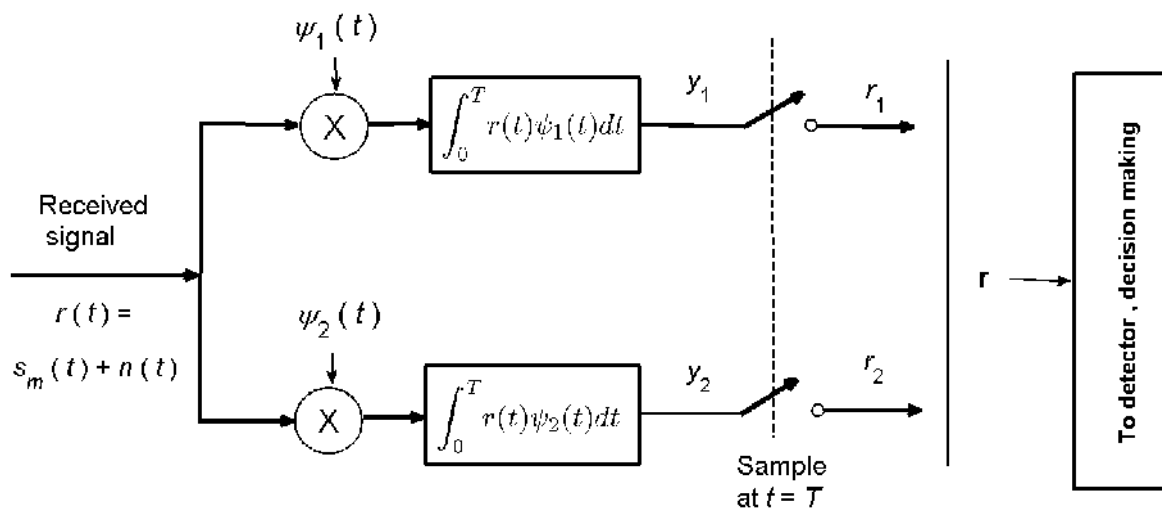
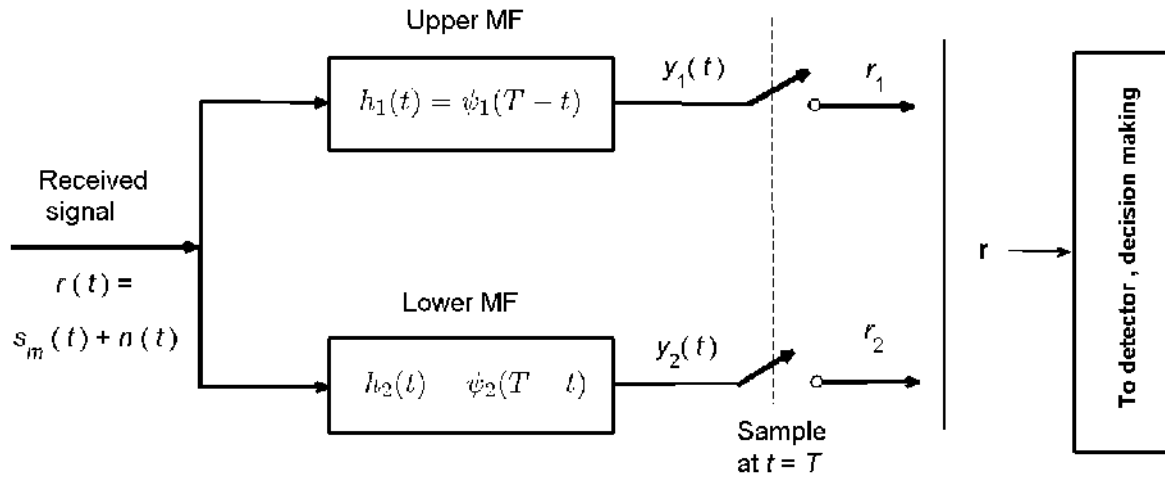


Fig. 6.3 Constellation diagram for the 4 PSK in Exercise 6.1.

Below, we show the block diagrams of correlator and matched filter type of demodulators.



a) Block diagram of correlator type of demodulator.



b) Block diagram of matched filter type of demodulator.

Fig. 6. 4 Block diagrams of correlator and matched filter type of demodulators for the 4 PSK in Exercise 6.1.

We first tackle the case of the correlator and assume that $s_1^a(t)$ was sent from the transmitter

$$\begin{aligned}
 y_1 = r_1 &= \int_0^T r(t) \psi_1(t) dt = \int_0^T s_1^a(t) \psi_1(t) dt + \int_0^T n(t) \psi_1(t) dt \\
 &= A\sqrt{T/2} + n_1, \quad n_1 = \int_0^T n(t) \psi_1(t) dt \\
 y_2 = r_2 &= \int_0^T r(t) \psi_2(t) dt = \int_0^T s_1^a(t) \psi_2(t) dt + \int_0^T n(t) \psi_2(t) dt \\
 &= A\sqrt{T/2} + n_2, \quad n_2 = \int_0^T n(t) \psi_2(t) dt, \quad \mathbf{r} = [r_1; r_2]
 \end{aligned} \tag{6.3}$$

The reason that we have arranged vector \mathbf{r} in the form of a column vector rather than a row vector, is because column arrangement facilitates metrics computation.

Doing the same for the matched filter case, we get

$$\begin{aligned}
 y_1(t) &= \int_0^t r(\tau) h_1(t-\tau) d\tau = \int_0^t r(\tau) \psi_1(T-t+\tau) d\tau \\
 y_2(t) &= \int_0^t r(\tau) h_2(t-\tau) d\tau = \int_0^t r(\tau) \psi_2(T-t+\tau) d\tau
 \end{aligned} \tag{6.4}$$

After sampling at $t = T$, (6.4) will become

$$r_1 = y_1(T) = \int_0^T r(\tau) \psi_1(\tau) d\tau = A\sqrt{T/2} + n_1, \quad n_1 = \int_0^T n(t) \psi_1(t) dt$$

$$r_2 = y_2(T) = \int_0^T r(\tau) \psi_2(\tau) d\tau = A\sqrt{T/2} + n_2, \quad n_2 = \int_0^T n(t) \psi_2(t) dt, \quad \mathbf{r} = [r_1; r_2] \quad (6.5)$$

So the outputs, we obtain from correlator and matched filter are the same.

Now we compute correlation metrics values by inserting $\mathbf{s}_1^a \dots \mathbf{s}_4^a$ into $C(\mathbf{r}, \mathbf{s}_m)$. Note that we can omit the common term at the end, that corresponds to the common energy of the signal vectors. Hence, from (6.3) and (6.5), we have

$$m = 1, \quad C(\mathbf{r}, \mathbf{s}_1) = 2 \mathbf{s}_1^a \cdot \mathbf{r} = 2 \left[A\sqrt{T/2}, A\sqrt{T/2} \right] \begin{bmatrix} A\sqrt{T/2} + n_1 \\ A\sqrt{T/2} + n_2 \end{bmatrix} = 2A^2T + An_1\sqrt{2T} + An_2\sqrt{2T}$$

$$m = 2, \quad C(\mathbf{r}, \mathbf{s}_2) = 2 \mathbf{s}_2^a \cdot \mathbf{r} = 2 \left[A\sqrt{T/2}, -A\sqrt{T/2} \right] \begin{bmatrix} A\sqrt{T/2} + n_1 \\ A\sqrt{T/2} + n_2 \end{bmatrix} = An_1\sqrt{2T} - An_2\sqrt{2T}$$

$$m = 3, \quad C(\mathbf{r}, \mathbf{s}_3) = 2 \mathbf{s}_3^a \cdot \mathbf{r} = 2 \left[-A\sqrt{T/2}, -A\sqrt{T/2} \right] \begin{bmatrix} A\sqrt{T/2} + n_1 \\ A\sqrt{T/2} + n_2 \end{bmatrix} = -2A^2T - An_1\sqrt{2T} - An_2\sqrt{2T}$$

$$m = 4, \quad C(\mathbf{r}, \mathbf{s}_4) = 2 \mathbf{s}_4^a \cdot \mathbf{r} = 2 \left[-A\sqrt{T/2}, A\sqrt{T/2} \right] \begin{bmatrix} A\sqrt{T/2} + n_1 \\ A\sqrt{T/2} + n_2 \end{bmatrix} = -An_1\sqrt{2T} + An_2\sqrt{2T} \quad (6.6)$$

In the presence of noise, to arrive at a correct decision, it must be that

$$C(\mathbf{r}, \mathbf{s}_1) > C(\mathbf{r}, \mathbf{s}_2), \quad C(\mathbf{r}, \mathbf{s}_1) > C(\mathbf{r}, \mathbf{s}_3), \quad C(\mathbf{r}, \mathbf{s}_1) > C(\mathbf{r}, \mathbf{s}_4) \quad (6.7)$$

These three conditions correspond to

$$C(\mathbf{r}, \mathbf{s}_1) > C(\mathbf{r}, \mathbf{s}_2) : A\sqrt{T/2} > -n_2 \rightarrow \overbrace{A\sqrt{T/2} > +n_2}^{r_2} > 0 \rightarrow r_2 > 0$$

$$C(\mathbf{r}, \mathbf{s}_1) > C(\mathbf{r}, \mathbf{s}_3) : \overbrace{A\sqrt{T/2} + n_1}^{r_1} > \overbrace{-A\sqrt{T/2} - n_2}^{-r_2} \rightarrow r_1 > -r_2$$

$$C(\mathbf{r}, \mathbf{s}_1) > C(\mathbf{r}, \mathbf{s}_4) : A\sqrt{T/2} > -n_1 \rightarrow \overbrace{A\sqrt{T/2} > +n_1}^{r_1} > 0 \rightarrow r_1 > 0 \quad (6.8)$$

The three conditions in (6.8) individually define the regions shown in Fig. 6.5.

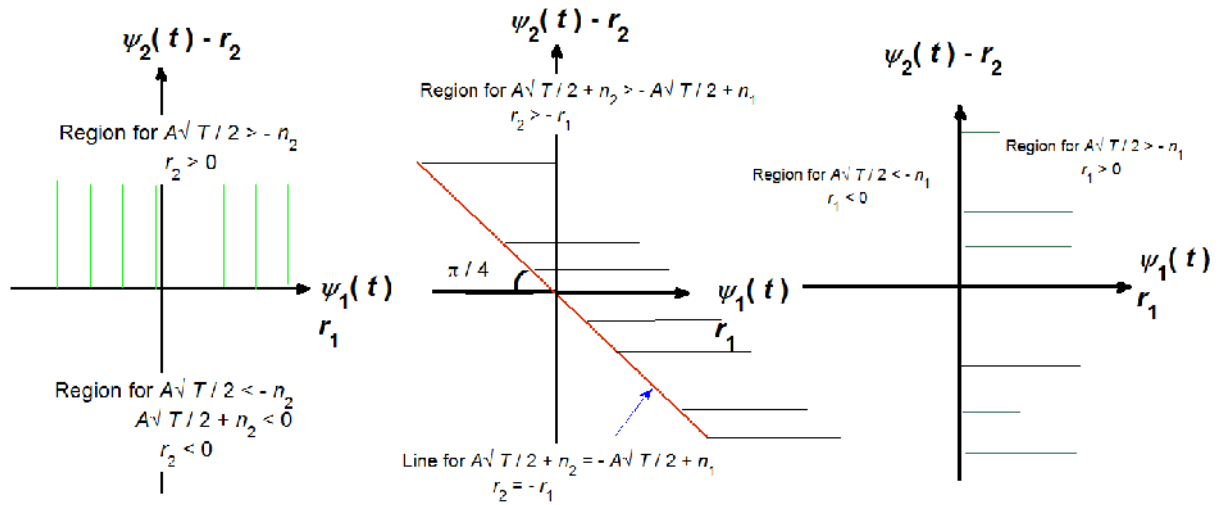


Fig. 6.5 Regions for conditions in (6.8).

Since conditions in (6.7) are connected by AND relation, then the intersection of the three regions of Fig. 6.5 becomes

The rest is left as an exercise. Furthermore assume $s_2^a(t)$ was sent from the transmitter and repeat the above.