Sample Problems – ECE 588 – ECE 632

Problem 7.19 of Proakis 2002 : The time waveforms of $s_1(t)\cdots s_3(t)$ are given as shown below. For these waveforms, find appropriate orthonormalized basis functions, find the representation of $s_1(t)\cdots s_3(t)$ in terms of orthonormalized basis functions, find $\mathbf{s}_1\cdots\mathbf{s}_3$ vectors, the distances between vector ends, draw constellation diagram and the diagram of demodulator comprising correlator and matched filter. Show that distance and correlation metrics function properly (i.e. give the correct decision) if $s_1(t)$ was sent from the transmitter and no noise is mixed with the signal at receiver. Give correct decision boundaries and find probability of error if all signals are sent from transmitter with equal probability.

Solution : $s_1(t)\cdots s_4(t)$ and two possible sets of $\psi_1(t),\ \psi_2(t)$ and $\psi_1^a(t),\ \psi_2^a(t)$ are given in Fig. 1.

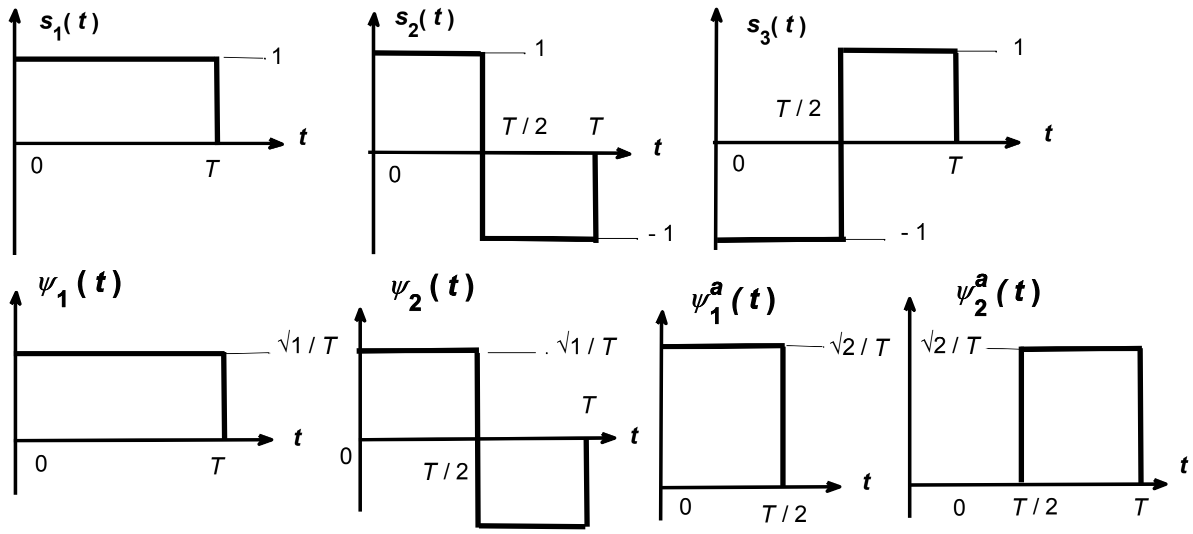


Fig. 1 Signal waveforms, $s_1(t)\cdots s_3(t)$ and orthonormalized basis functions, $\psi_1(t),\ \psi_2(t)$ and $\psi_1^a(t),\ \psi_2^a(t)$ for the given question.

From Fig. 1, we understand that the signal set $s_1(t)\cdots s_3(t)$ is two dimensional, that is $N=2$ and $M=3$. For the representation of $s_1(t)\cdots s_3(t)$, we can choose either $\psi_1(t),\ \psi_2(t)$ or $\psi_1^a(t),\ \psi_2^a(t)$. Both are given below.

$$s_1(t)=\sqrt{T}\psi_1(t) \ , \ s_2(t)=\sqrt{T}\psi_2(t) \ , \ s_3(t)=-\sqrt{T}\psi_2(t)$$
$$\mathbf{s}_1=[s_{11},\ s_{12}]=\left[\sqrt{T},\ 0\right] \ , \ \mathbf{s}_2=[s_{21},\ s_{22}]=\left[0,\ \sqrt{T}\right] \ , \ \mathbf{s}_3=[s_{31},\ s_{32}]=\left[-\sqrt{T},\ 0\right]$$
$$s_1(t)=\sqrt{\frac{T}{2}}\psi_1^a(t)+\sqrt{\frac{T}{2}}\psi_2^a(t) \ , \ s_2(t)=\sqrt{\frac{T}{2}}\psi_1^a(t)-\sqrt{\frac{T}{2}}\psi_2^a(t) \ , \ s_3(t)=-\sqrt{\frac{T}{2}}\psi_1^a(t)+\sqrt{\frac{T}{2}}\psi_2^a(t)$$
$$\mathbf{s}_1=[s_{11},\ s_{12}]=\left[\sqrt{\frac{T}{2}},\ \sqrt{\frac{T}{2}}\right] \ , \ \mathbf{s}_2=[s_{21},\ s_{22}]=\left[\sqrt{\frac{T}{2}},\ -\sqrt{\frac{T}{2}}\right] \ , \ \mathbf{s}_3=[s_{31},\ s_{32}]=\left[-\sqrt{\frac{T}{2}},\ \sqrt{\frac{T}{2}}\right]$$
$$d_{12}=d_{13}=\sqrt{2T} \ , \ d_{23}=2\sqrt{T}$$
$$|\mathbf{s}_1|=|\mathbf{s}_2|=|\mathbf{s}_3|=\sqrt{T} \ , \ \varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_s=T \qquad (1)$$

As understood from the last line of (1), the given signal set of $s_1(t)\cdots s_3(t)$ constitutes 3 PSK.

The constellation diagrams with respect to $\psi_1(t),\ \psi_2(t)$ and $\psi_1^a(t),\ \psi_2^a(t)$ are plotted below.

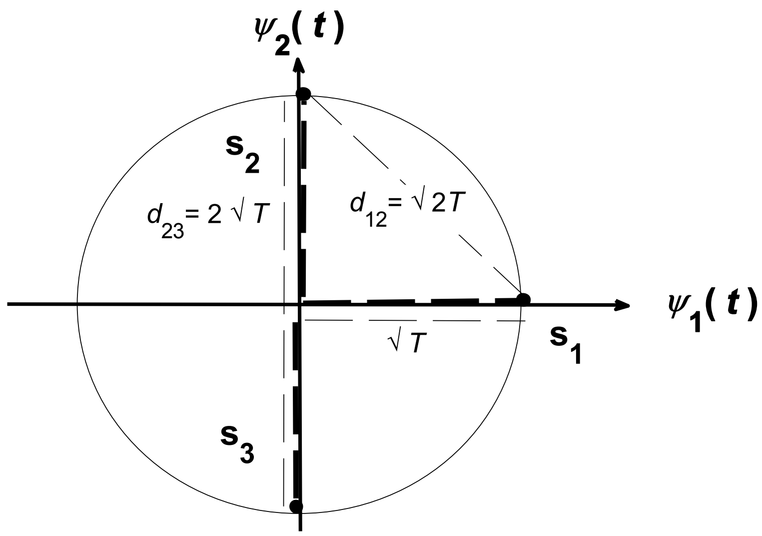


Fig. 2 Constellation diagram for the 3 PSK with respect to $\psi_1(t),\ \psi_2(t)$.

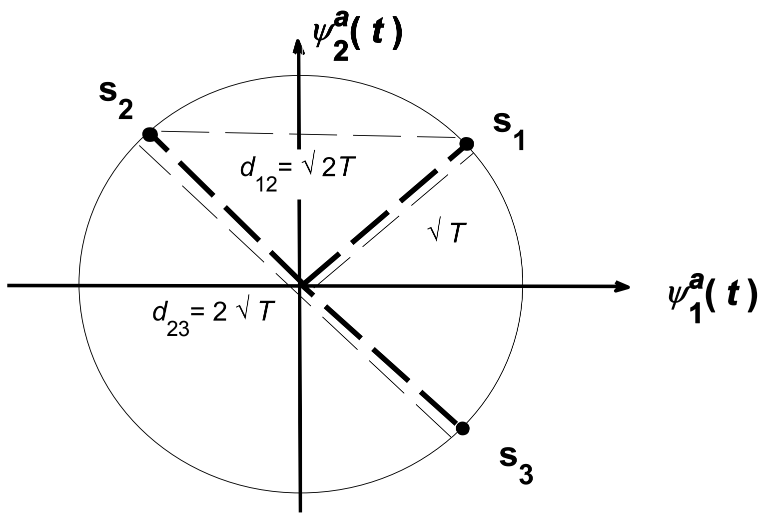


Fig. 3 Constellation diagram for the 3 PSK with respect to $\psi_1^a(t),\ \psi_2^a(t)$.

Both in Figs. 2 and 3, the signal space diagram is used inefficiently. An optimum constellation is given in Fig. 4, where the angular spacing between signal vectors are 120^0 and the distance between signal vector ends are equivalent and maximized.

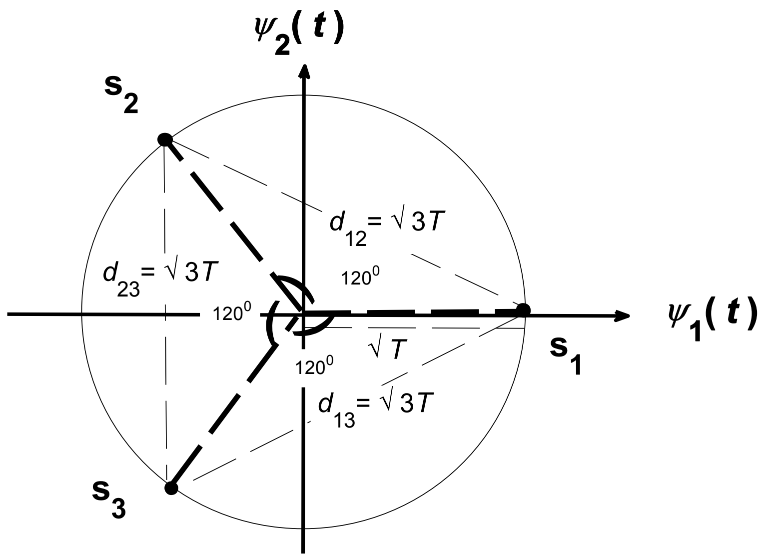
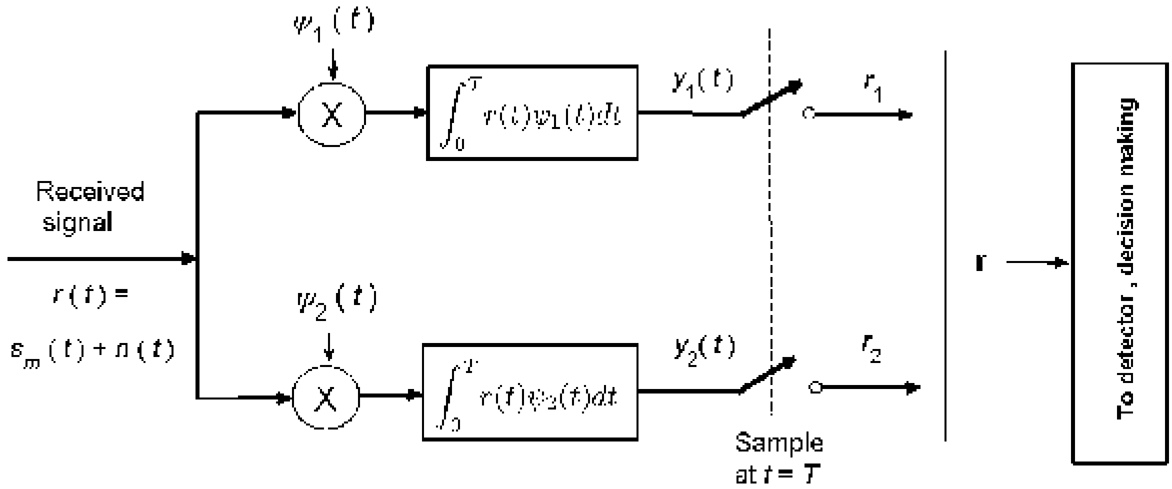
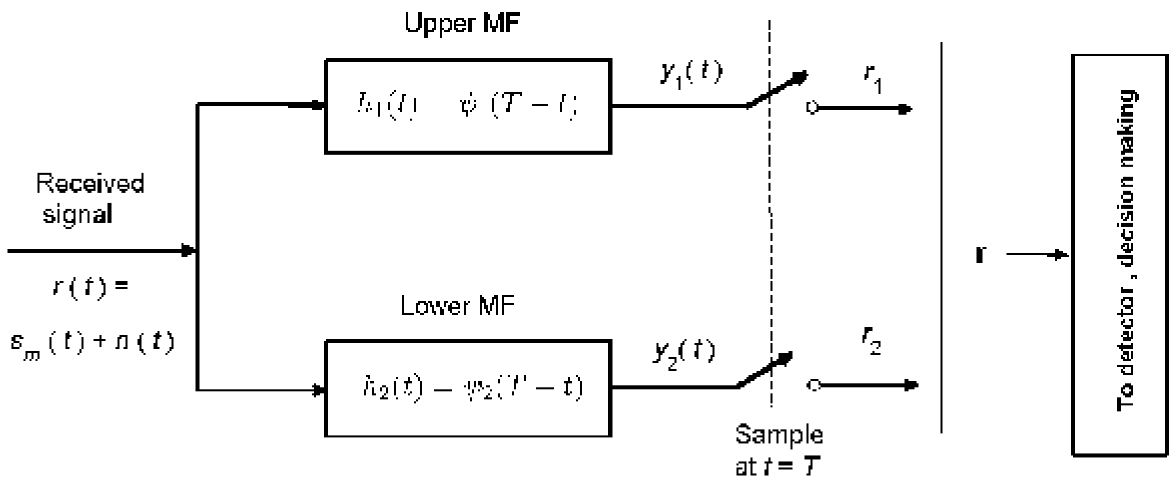


Fig. 3 An optimum constellation diagram for the 3 PSK case.

For the analysis below, we proceed with $\psi_1(t),\ \psi_2(t)$. The block diagrams of correlator and matched filter type of demodulators are shown below.



a) Block diagram of correlator type of demodulator.



b) Block diagram of matched filter type of demodulator.

Fig. 4 Block diagrams of correlator and matched filter type of demodulators for the 3 PSK in the given question.

To arrive at probability of error and decision boundaries, we examine the cases $s_1(t)$ and $s_2(t)$ being sent from the transmitter, since the locations of $\mathbf{s}_2$ and $\mathbf{s}_3$ are interchangable.

In this context, we evaluate the sampled outputs of the correlator or matched filter in the cases of $s_1(t)$ and $s_2(t)$ being sent from the transmitter. From Example 6.2 of the document entitled, "Notes in Dimensionality of Signals_Sept 2012", it is easy to see that, the $\mathbf{r}$ vector will be

$$\mathbf{r}_{\mathbf{s}_1} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \sqrt{T} + n_1 \\ n_2 \end{bmatrix} \quad \text{if } s_1(t) \text{ is sent from transmitter}$$

$$\mathbf{r}_{\mathbf{s}_2} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} n_1 \\ \sqrt{T} + n_2 \end{bmatrix} \quad \text{if } s_2(t) \text{ is sent from transmitter} \qquad (2)$$

Due to randomness of noise n_1 and n_2 of the first and second lines are actually different. Below we calculate the correlation metrics for both cases.

$$C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_m) = 2\,\mathbf{s}_m \cdot \mathbf{r}_{\mathbf{s}_1} - \|\mathbf{s}_m\|^2 \quad \text{if } s_1(t) \text{ is sent from transmitter}$$

$$m = 1,\ C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_1) = 2\,\mathbf{s}_1 \cdot \mathbf{r}_{\mathbf{s}_1} - \|\mathbf{s}_1\|^2 = 2\left[\sqrt{T},\ 0\right] \begin{bmatrix} \sqrt{T} + n_1 \\ n_2 \end{bmatrix} - T = 2n_1\sqrt{T} + T$$

$$m = 2,\ C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_2) = 2\,\mathbf{s}_2 \cdot \mathbf{r}_{\mathbf{s}_1} - \|\mathbf{s}_2\|^2 = 2\left[0,\ \sqrt{T}\right] \begin{bmatrix} \sqrt{T} + n_1 \\ n_2 \end{bmatrix} - T = 2n_2\sqrt{T} - T$$

$$m = 3,\ C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_3) = 2\,\mathbf{s}_3 \cdot \mathbf{r}_{\mathbf{s}_1} - \|\mathbf{s}_3\|^2 = 2\left[0,\ -\sqrt{T}\right] \begin{bmatrix} \sqrt{T} + n_1 \\ n_2 \end{bmatrix} - T = -2n_2\sqrt{T} - T \qquad (3a)$$

$$C(\mathbf{r}_{\mathbf{s}_2}, \mathbf{s}_m) = 2\,\mathbf{s}_m \cdot \mathbf{r}_{\mathbf{s}_2} - \|\mathbf{s}_m\|^2 \quad \text{if } s_2(t) \text{ is sent from transmitter}$$

$$m = 1,\ C(\mathbf{r}_{\mathbf{s}_2}, \mathbf{s}_1) = 2\,\mathbf{s}_1 \cdot \mathbf{r}_{\mathbf{s}_1} - \|\mathbf{s}_1\|^2 = 2\left[\sqrt{T},\ 0\right] \begin{bmatrix} n_1 \\ \sqrt{T} + n_2 \end{bmatrix} - T = 2n_1\sqrt{T} - T$$

$$m = 2,\ C(\mathbf{r}_{\mathbf{s}_2}, \mathbf{s}_2) = 2\,\mathbf{s}_2 \cdot \mathbf{r}_{\mathbf{s}_1} - \|\mathbf{s}_2\|^2 = 2\left[0,\ \sqrt{T}\right] \begin{bmatrix} n_1 \\ \sqrt{T} + n_2 \end{bmatrix} - T = 2n_2\sqrt{T} + T$$

$$m = 3,\ C(\mathbf{r}_{\mathbf{s}_2}, \mathbf{s}_3) = 2\,\mathbf{s}_3 \cdot \mathbf{r}_{\mathbf{s}_1} - \|\mathbf{s}_3\|^2 = 2\left[0,\ -\sqrt{T}\right] \begin{bmatrix} n_1 \\ \sqrt{T} + n_2 \end{bmatrix} - T = -2n_2\sqrt{T} - 3T \qquad (3b)$$

To arrive at a correct decision by the detector, in (3a), that is if $s_1(t)$ is sent from the transmitter, the following condition should hold

$$C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_1) > C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_2),\ C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_1) > C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_3) \qquad (4)$$

We observe that (3a) and (4) are quite similar to (6.23) and (6.24) of Notes in Dimensionality of Signals_Sept 2012, except that $\mathbf{s}_4$ is missing here or $\mathbf{s}_4$ of the Notes in Dimensionality of Signals_Sept 2012 document is replaced here by $\mathbf{s}_3$.

The two conditions in (4) correspond to

$$C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_1) > C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_2) : T + 2n_1\sqrt{T} > 2n_2\sqrt{T} - T \ \rightarrow \sqrt{T} + n_1 > n_2$$

$$C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_1) > C(\mathbf{r}_{\mathbf{s}_1}, \mathbf{s}_3) : T + 2n_1\sqrt{T} > -2n_2\sqrt{T} - T \ \rightarrow \ \sqrt{T} + n_1 > -n_2 \qquad (5)$$

The regions defined by the two conditions of (5) are shown below in Fig. 5

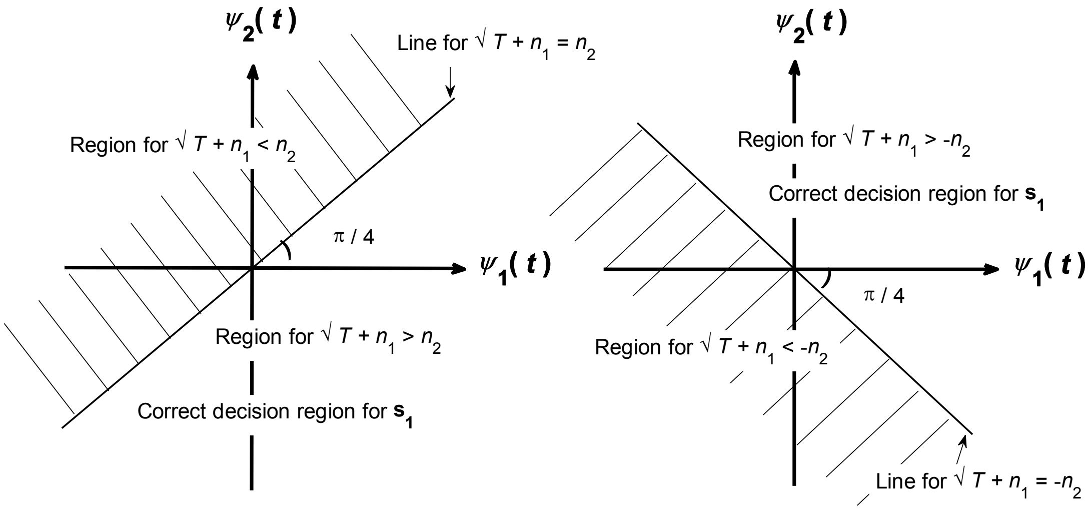


Fig. 5 Regions for conditions in (5).

So the pictorial representation indicates that

$$\left|\frac{n_2}{\sqrt{T}+n_1}\right| \leq 1 \quad \text{or} \quad \operatorname{atan}\left(\frac{n_2}{\sqrt{T}+n_1}\right) \leq \left|\frac{\pi}{4}\right| \tag{6}$$

The condition set out in (6) is incomplete, since we may have the situation depicted in Fig. 6

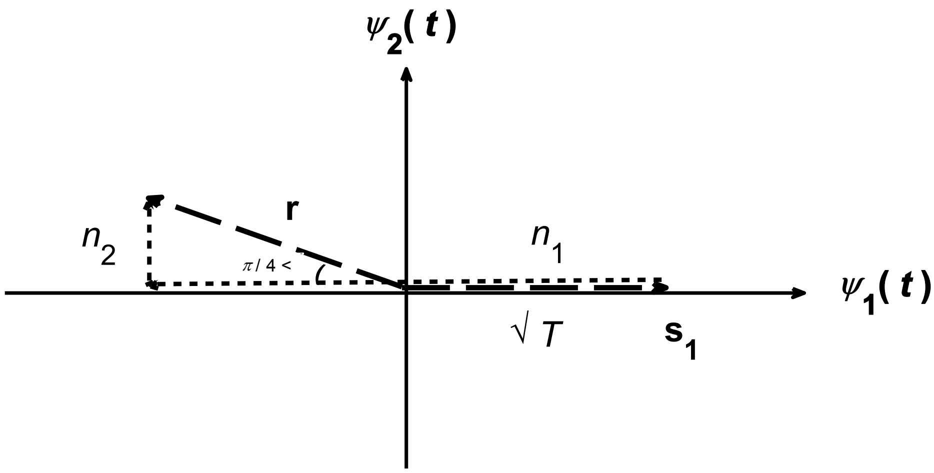


Fig. 6 The case of $-n_1 > \sqrt{T}$.

Fig. 6 demonstrates that as an addition to (6), we must have vector laying along the positive axis of $\psi_1(t)$, which means that $\sqrt{T} > -n_1$. With this addition, we arrive at exactly the set defined in Notes in Dimensionality of Signals_Sept 2012 document for 4 PSK case. This is not surprising since the locations of adjacent signal vectors of the 3 PSK case examined here are exactly the same as 4 PSK case.

If we now return to $s_2(t)$ being sent from the transmitter, we have from (3b)

$$C(\mathbf{r}_{\mathbf{s}_2}, \mathbf{s}_2) > C(\mathbf{r}_{\mathbf{s}_2}, \mathbf{s}_1) \; : 2n_2\sqrt{T} - T > 2n_1\sqrt{T} - T \;\; \rightarrow \sqrt{T} + n_2 > n_1$$
$$C(\mathbf{r}_{\mathbf{s}_2}, \mathbf{s}_2) > C(\mathbf{r}_{\mathbf{s}_2}, \mathbf{s}_3) \; : 2n_1\sqrt{T} + T > -2n_2\sqrt{T} - 3T \;\; \rightarrow \; \sqrt{T} > -n_2 \qquad (7)$$

The regions defined by the two conditions of (7) are shown below in Fig. 7

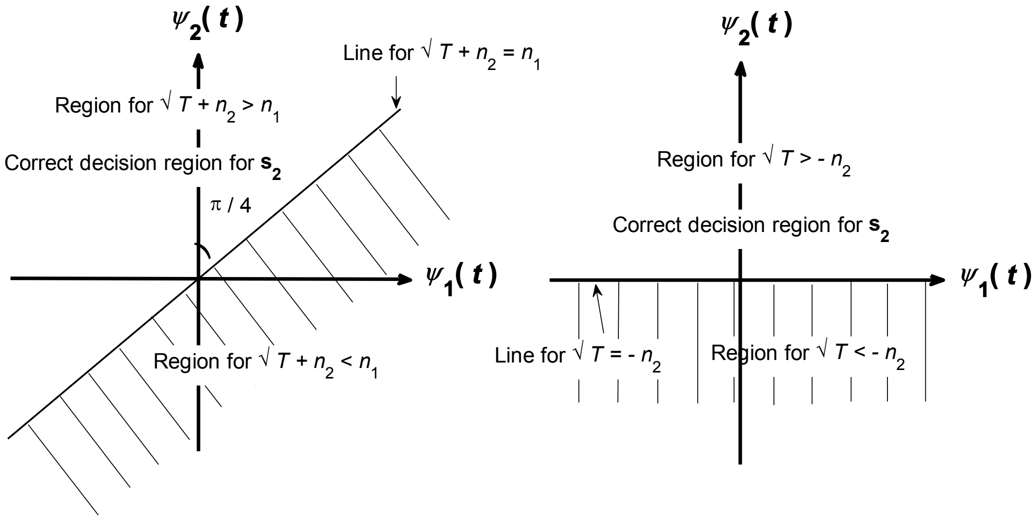


Fig. 7 Regions for conditions in (7).

By considering the intersection of correct decision regions in Figs. 5 and 7, we find the following decision regions for $\mathbf{s}_1, \; \mathbf{s}_2$ and $\mathbf{s}_3$

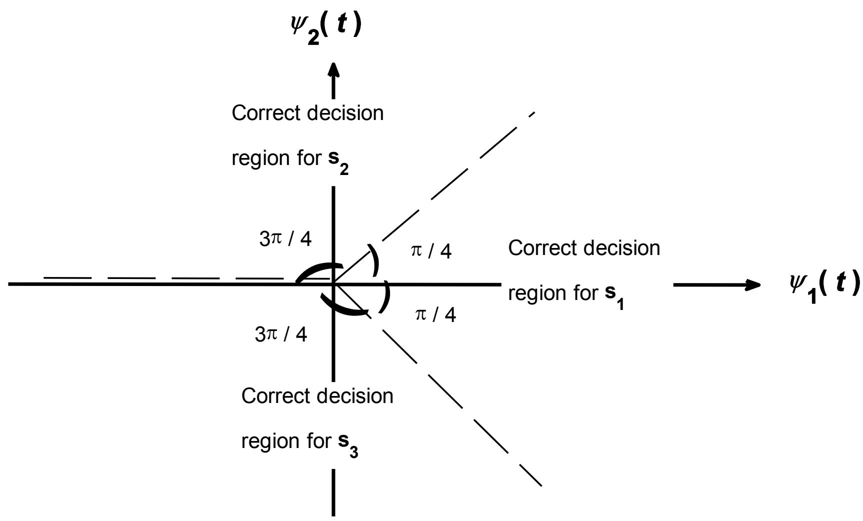


Fig. 8 Correct decision regions for $\mathbf{s}_1, \; \mathbf{s}_2$ and $\mathbf{s}_3$.

According to the regions defined in Fig. 8 and based on (6.28) of the Notes in Dimensionality of Signals_Sept 2012 document, we can write the probability of correct decision for $\mathbf{s}_1$ as

$$P_c = \frac{1}{(\pi N_0)^{0.5}} \int_{-\sqrt{T}}^{\infty} \exp\left(-\frac{n_1^2}{N_0}\right) dn_1 \frac{1}{(\pi N_0)^{0.5}} \int_{-(n_1+\sqrt{T})\tan\left(\frac{\pi}{4}\right)}^{(n_1+\sqrt{T})\tan\left(\frac{\pi}{4}\right)} \exp\left(-\frac{n_2^2}{N_0}\right) dn_2 \qquad \text{for } \mathbf{s}_1 \qquad (8)$$

whereas $\mathbf{s}_2$ and $\mathbf{s}_3$, the probability of correct decision will be

$$P_c = \frac{1}{(\pi N_0)^{0.5}} \int_{-\sqrt{T}}^{\infty} \exp\left(-\frac{n_2^2}{N_0}\right) dn_2 \frac{1}{(\pi N_0)^{0.5}} \int_{-\infty}^{(n_2+\sqrt{T})\tan\left(\frac{\pi}{4}\right)} \exp\left(-\frac{n_1^2}{N_0}\right) dn_1 \qquad \text{for } \mathbf{s}_2 \text{ and } \mathbf{s}_3 \qquad (9)$$

It is clear form (8) and (9), the first integrals are identical, but when the second integrals into account, the following will apply

$$P_c(s_1) < P_c(\mathbf{s}_2 \text{ and } \mathbf{s}_3) \text{ or } P_e(\mathbf{s}_1) > P_e(\mathbf{s}_2 \text{ and } \mathbf{s}_3) \;, \quad P_e = 1 - P_c \qquad (10)$$

The limits of the integration in the second integrals of (8) and (9) are shown in Fig. 9. The graph of the Gaussian exponential for n_1 is similar to the case of n_2 illustrated in Fig. 9.

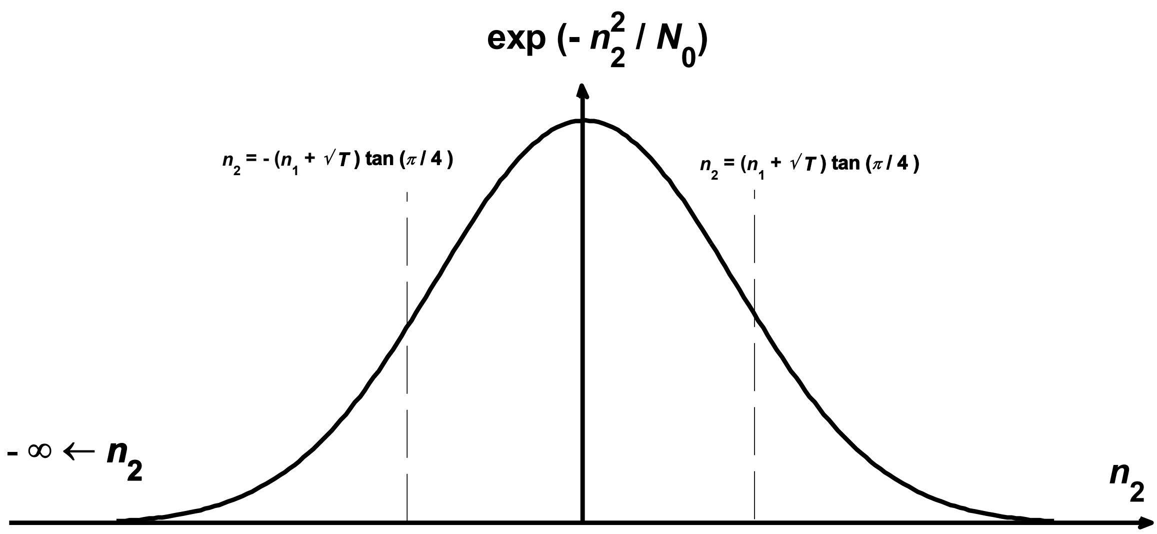


Fig. 8 Illustration for the limits of integrations of the second integrals in (8) and (9).