Çankaya University – ECE Department – ECE 632 (MT)

Student Name : Date : 20.11.2012
Student Number : Open Source Exam

Questions

1. (70 Points) Three signals (time waveforms) $s_1(t) \cdots s_3(t)$ are given as shown below.

   a. Find and plot the orthonormal basis functions of this signal set, 1) By working through Gram-Schmidt orthogonalization procedure or by eye inspection and 2) By using the m file, ECE632_GSOrthogonalWavefor_Exp1.m available on the ECE632 course webpage. Check that the two results agree. When representing the triangular waveforms of $s_1(t) \cdots s_3(t)$ in MATLAB, initially set A and T to numeric values, then use the principle of sampling to write for $s_1(t) \cdots s_3(t)$. This way, if $A = 1,\ T = 10$, then in MATLAB notation, we would have the eleven samples of $s_1(t)$ as $s_1(t) = [0,\ 1/10,\ 2/10,\ 3/10,\ 4/10,\ 5/10,\ 6/10,\ 7/10,\ 8/10,\ 9/10,\ 10/10]$.
   b. Identify the type of modulation and dimensionality in this set. Draw the constellation diagram, the signal vectors $\mathbf{s}_1 \cdots \mathbf{s}_3$, indicating the length of signal vectors and distances between the vector ends.
   c. Draw the demodulator as correlator and matched filter. Assuming that $s_1(t)$ was transmitted, find the outputs of the correlator and matched filter.
   d. Find the probability of error and decision regions via the evaluations of correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ again assuming $s_1(t)$ was transmitted.

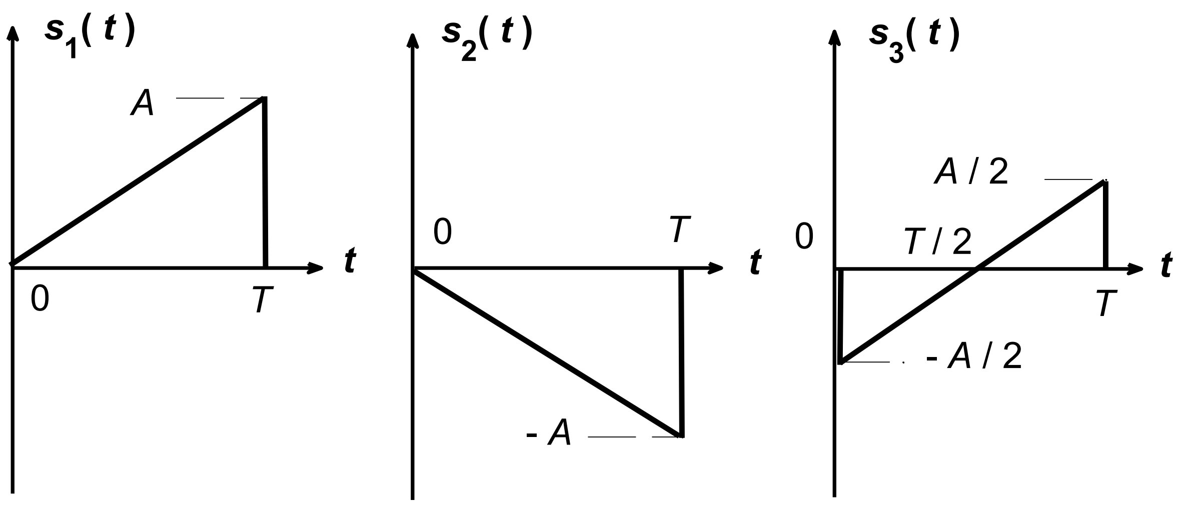


Fig. 1 Time waveforms given in question Q1.

Solution : a. We write the mathematical expressions for the time waveforms of $s_1(t) \cdots s_3(t)$ as follows

$$s_1(t)=\begin{cases} A\dfrac{t}{T} & 0\le t\le T \\ 0 & \text{elsewhere}\end{cases} \qquad s_2(t)=\begin{cases} -A\dfrac{t}{T} & 0\le t\le T \\ 0 & \text{elsewhere}\end{cases} \qquad s_3(t)=\begin{cases} \dfrac{A}{T}\left(t-\dfrac{T}{2}\right) & 0\le t\le T \\ 0 & \text{elsewhere}\end{cases} \tag{1}$$

Since $s_1(t)=-s_2(t)$, the basis function $\psi_1(t)$ can simply be set to normalized version of $s_1(t)$ or $s_2(t)$. Here we choose $s_1(t)$, thus $\psi_1(t)$ will be

$$\psi_1(t)=\frac{s_1(t)}{\sqrt{\varepsilon_{s_1}}}=\frac{1}{T}\sqrt{\frac{3}{T}}t \tag{2}$$

Now we proceed to find $\psi_2(t)$, which according to Gram Schmidt orthogonalization steps will be given as

$$\psi_2(t)=\frac{d_2(t)}{\sqrt{\varepsilon_{d_2}}}\ ,\quad \varepsilon_{d_2}=\int_0^T d_2^2(t)\,dt\ ,\quad d_2(t)=s_3(t)-c_{21}\psi_1(t)\ ,\quad c_{21}=\int_0^T s_3(t)\ \psi_1(t)dt \tag{3}$$

Evaluating the quantities in (3), we find that

$$\varepsilon_{d_2}=\frac{A^2T}{16}\ ,\ c_{21}=A\sqrt{\frac{T}{3}}-\frac{A\sqrt{3T}}{4}\ ,\quad d_2(t)=\frac{3A}{4T}t-\frac{A}{2} \tag{4}$$

So eventually we get

$$\psi_2(t)=\frac{d_2(t)}{\sqrt{\varepsilon_{d_2}}}=\frac{3}{T\sqrt{T}}t-\frac{2}{\sqrt{T}} \tag{5}$$

It is easy to test that

$$\int_0^T \psi_1^2(t)\,dt=1\ ,\ \int_0^T \psi_2^2(t)\,dt=1\ ,\ \int_0^T \psi_1(t)\ \psi_2(t)dt=0 \tag{6}$$

So $\psi_1(t)$ and $\psi_2(t)$ satisfy the conditions of carrying unit energy and being orthogonal to each other. The plots of $\psi_1(t)$ and $\psi_2(t)$ are shown in Fig. 2.

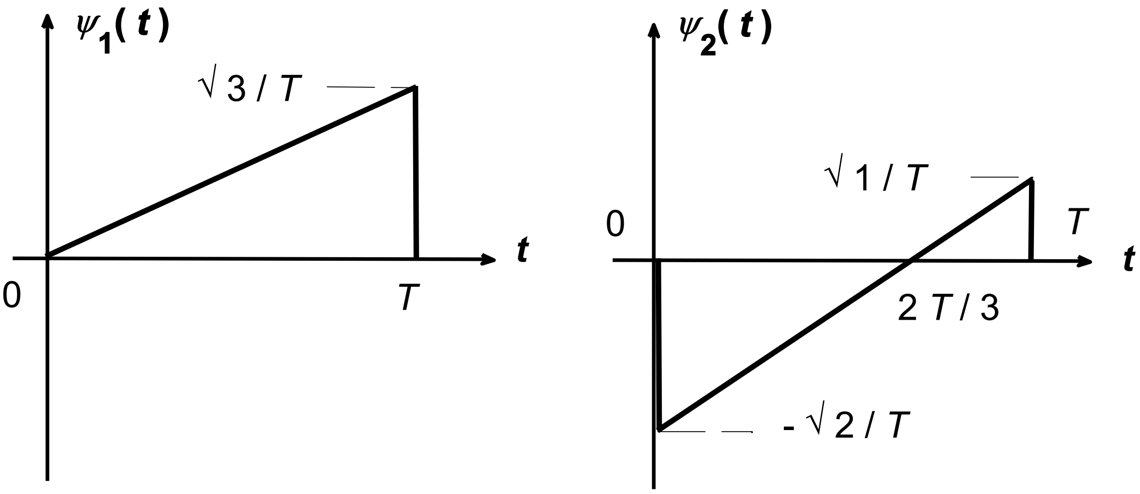


Fig. 2 Orthonormal basis functions for the signal set given in the question Q1.

Adopting the case of eleven samples given in the question, the MATLAB representation of our signals, $s_1(t)\cdots s_3(t)$ will be

$$s(t)=\begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix}\begin{bmatrix} 0, & 1/10, & 2/10, & 3/10, & 4/10, & 5/10, & 6/10, & 7/10, & 8/10, & 9/10, & 10/10 \\ 0, & -1/10, & -2/10, & -3/10, & -4/10, & -5/10, & -6/10, & -7/10, & -8/10, & -9/10, & -10/10 \\ -5/10, & -4/10, & -3/10, & -2/10, & -1/10, & 0, & 1/10, & 2/10, & 3/10, & 4/10, & 5/10 \end{bmatrix} \quad (7)$$

With the representation in (7), it is possible to obtain pretty close numeric values, when we compare the Gram Schmidt solution of the above and that of ECE632_GSOrthogonalWavefor_Exp1.m. Obtaining agreement between the two cases up to 4 digit accuracy is only possible by going to higher sample sizes. This case is shown in the m file of ECE632_MTQ120112012.m.

b. Since there are two basis functions, this is two dimensional modulation type, so either PSK or QAM. Finding the energies of $s_1(t)\cdots s_3(t)$, we get

$$\varepsilon_{s_1}=\int_0^T s_1^2(t)\,dt=\varepsilon_{s_2}=\int_0^T s_1^2(t)\,dt=\frac{A^2T}{3}\ ,\quad \varepsilon_{s_3}=\frac{A^2T}{12}\ ,\quad \varepsilon_{s_1}=\varepsilon_{s_2}\neq\varepsilon_{s_3}\ ,\quad \varepsilon_{s_1}=\varepsilon_{s_2}>\varepsilon_{s_3} \quad (8)$$

So this is QAM. Note that the reason why the energies in $s_1(t)$ and $s_2(t)$ are greater than the energy in $s_3(t)$ is that $s_1(t)$ and $s_2(t)$ contain DC energy, whereas this is not the case for $s_3(t)$, since $s_3(t)$ is symmetrical about zero amplitude. To plot the constellation diagram, we must find the vectorial components of $\mathbf{s}_1\cdots\mathbf{s}_3$. The cases of $\mathbf{s}_1$ and $\mathbf{s}_2$ will be quite simple, since they will only have components along $\psi_1(t)$ such that

$$\mathbf{s_1} = [s_{11},\ s_{12}]\ ,\ s_{11} = \int_0^T s_1(t)\ \psi_1(t) dt = A\sqrt{\frac{T}{3}}\ ,\ s_{12} = \int_0^T s_1(t)\ \psi_2(t) dt = 0$$

$$\mathbf{s_2} = [s_{21},\ s_{22}]\ ,\ s_{21} = \int_0^T s_2(t)\ \psi_1(t) dt = -A\sqrt{\frac{T}{3}}\ ,\ s_{22} = \int_0^T s_2(t)\ \psi_2(t) dt = 0 \quad (9)$$

But $\mathbf{s_3}$ will have components both along $\psi_1(t)$ and $\psi_2(t)$. If we write $\mathbf{s_3}$ as $\mathbf{s_3} = [s_{31},\ s_{32}]$, then from (3) and (4), we get

$$s_{31} = c_{21} = A\sqrt{\frac{T}{3}} - \frac{A\sqrt{3T}}{4} \quad (10)$$

And we undertake the following calculations for s_{32}

$$s_{32} = \int_0^T s_3(t)\ \psi_2(t) dt = \int_0^T \frac{A}{T}\left(t - \frac{T}{2}\right)\left(\frac{3}{T\sqrt{T}} t - \frac{2}{\sqrt{T}}\right) dt = \frac{A}{4}\sqrt{T} \quad (11)$$

Hence $\mathbf{s_3}$ will be

$$\mathbf{s_3} = [s_{31},\ s_{32}] = \left[A\sqrt{\frac{T}{3}} - \frac{A\sqrt{3T}}{4},\ \frac{A}{4}\sqrt{T}\right]\ ,\ \|\mathbf{s_3}\|^2 = \frac{A^2 T}{12} \quad (12)$$

Comparing (12) with (8), it is easy to see that $\|\mathbf{s_3}\|^2 = \varepsilon_{s_3}$. In Fig. 3, we show the constellation diagram together with distances between vector ends.

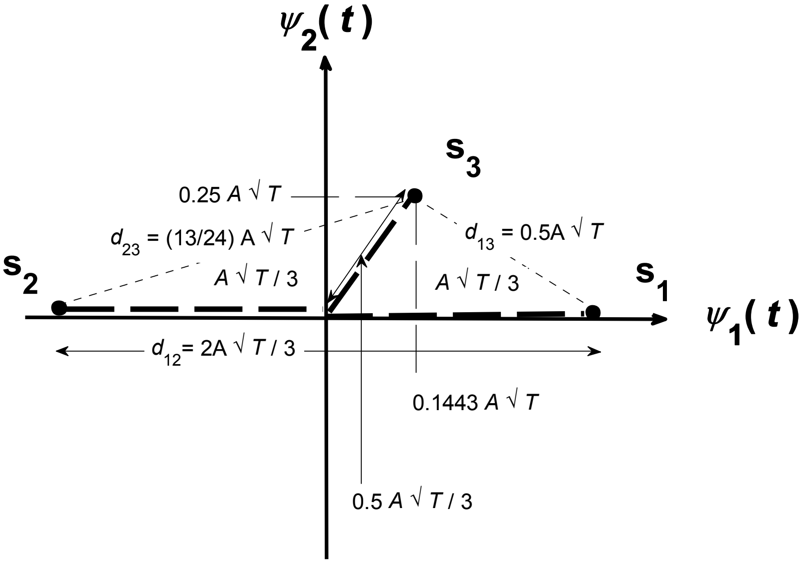
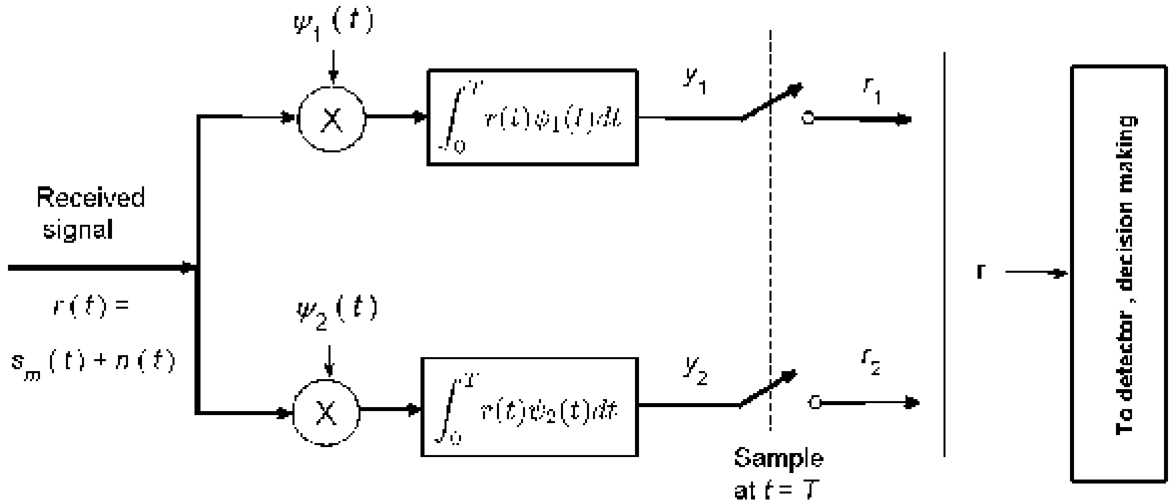
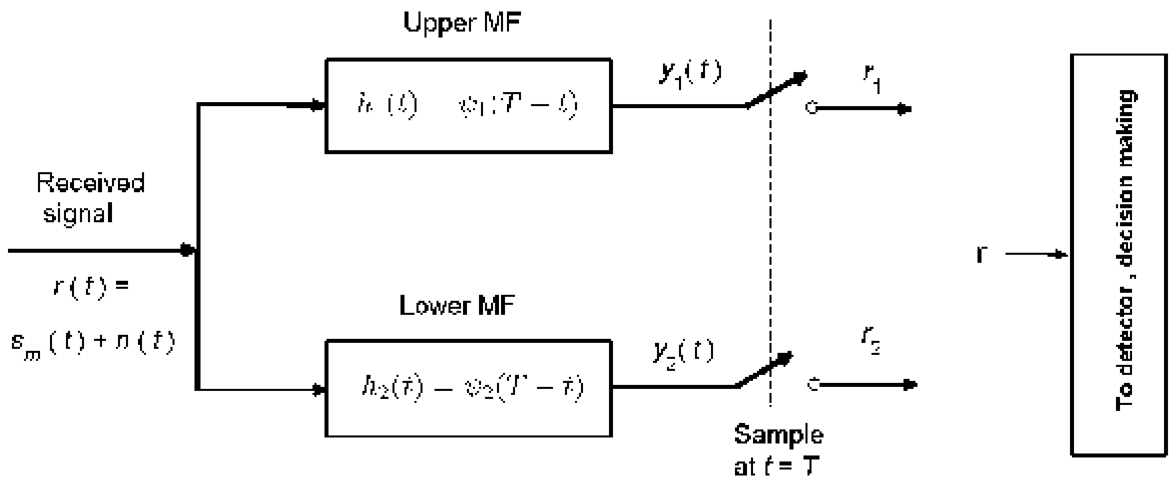


Fig. 3 Constellation diagram of the signal set given in the question Q1.

c. Since QAM and PSK are both two dimensional, for block diagrams of correlator and MF, we benefit from Fig. 6.7 of Notes on Dimensionality of Signals_Sept 2012_HTE, which we reproduce below in Fig. 4



a) Block diagram of correlator type of demodulator.



b) Block diagram of matched filter (MF) type of demodulator.

Fig. 4 Block diagrams of correlator and matched filter type of demodulators for the signal set in Q1.

If $s_1(t)$ is sent from the transmitter, then by benefiting from upper line of (9), the output from the upper and lower braches of the correlator will be (before or after sampling)

$$\begin{aligned} y_1 &= \int_0^T r(t)\ \psi_1(t)dt = \int_0^T s_1(t)\ \psi_1(t)dt + \int_0^T n(t)\ \psi_1(t)dt = s_{11} + n_1 = A\sqrt{\frac{T}{3}} + n_1 \\ y_2 &= \int_0^T r(t)\ \psi_2(t)dt = \int_0^T s_1(t)\ \psi_2(t)dt + \int_0^T n(t)\ \psi_2(t)dt = s_{12} + n_2 = 0 + n_2 \end{aligned} \quad (13)$$

We know that output from MF will be identical to (12) at the time of sampling at $t = T$ which means that we can construct the received vector $\mathbf{r}$ that we supply to the detector and which will be used in the decision making process, as follows

$$y_1 = y_1(t=T) = A\sqrt{\frac{T}{3}} + n_1 = r_1 \ , \ y_2 = y_2(t=T) = 0 + n_2 = r_2$$

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} A\sqrt{\frac{T}{3}} + n_1 \\ n_2 \end{bmatrix} \tag{14}$$

The constellation describing (13) is found in Fig. 5.

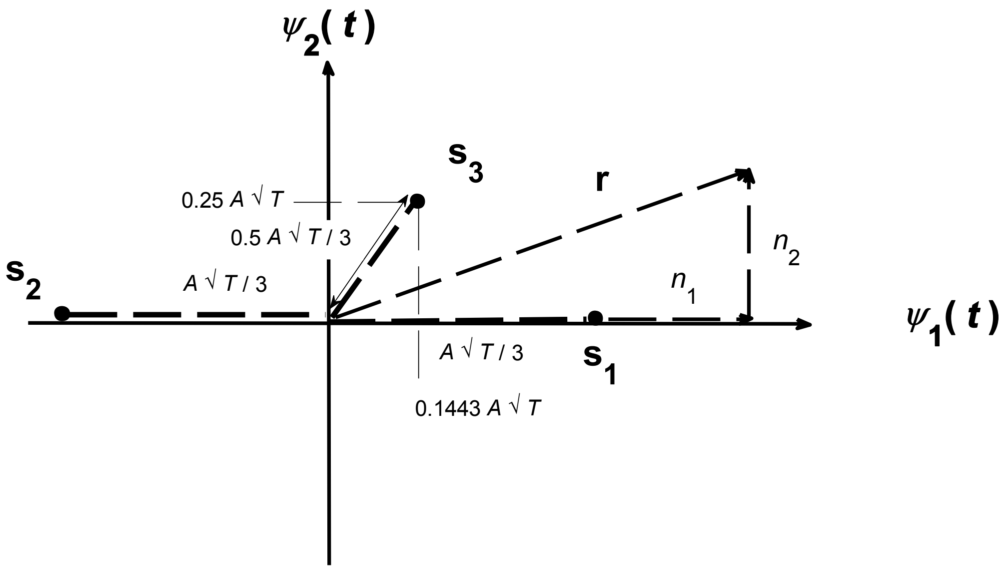


Fig. 5 The constellation when $s_1(t)$ is sent from the transmitter in Q1.

d. In order to arrive at probability of error and define decision regions (for the case of $s_1(t)$ being sent from transmitter), from (13) we evaluate correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$, thus

$$C(\mathbf{r}, \mathbf{s}_m) = 2\,\mathbf{s}_m \cdot \mathbf{r} - \|\mathbf{s}_m\|^2 \ , \quad m = 1 \cdots 3$$

$$m=1 \ , \ C(\mathbf{r}, \mathbf{s}_1) = 2\,\mathbf{s}_1 \cdot \mathbf{r} - \|\mathbf{s}_1\|^2 = 2\left[A\sqrt{\frac{T}{3}}, \ 0\right]\begin{bmatrix} A\sqrt{\frac{T}{3}} + n_1 \\ n_2 \end{bmatrix} - \frac{A^2T}{3} = \frac{A^2T}{3} + 2An_1\sqrt{\frac{T}{3}}$$

$$m=2 \ , \ C(\mathbf{r}, \mathbf{s}_2) = 2\,\mathbf{s}_2 \cdot \mathbf{r} - \|\mathbf{s}_2\|^2 = 2\left[-A\sqrt{\frac{T}{3}}, \ 0\right]\begin{bmatrix} A\sqrt{\frac{T}{3}} + n_1 \\ n_2 \end{bmatrix} - \frac{A^2T}{3} = -A^2T - 2An_1\sqrt{\frac{T}{3}}$$

$$m=3 \ , \ C(\mathbf{r}, \mathbf{s}_3) = 2\,\mathbf{s}_3 \cdot \mathbf{r} - \|\mathbf{s}_3\|^2 = 2\left[A\sqrt{\frac{T}{3}} - \frac{A\sqrt{3T}}{4}, \ \frac{A}{4}\sqrt{T}\right]\begin{bmatrix} A\sqrt{\frac{T}{3}} + n_1 \\ n_2 \end{bmatrix} - \frac{A^2T}{12}$$

$$= \frac{A^2T}{12} + An_1\sqrt{T}\left(\frac{2}{\sqrt{3}} - \frac{\sqrt{3}}{2}\right) + \frac{An_2\sqrt{T}}{2} \tag{15}$$

To arrive at a correct decision, the following conditions must hold

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_2) : \frac{A^2T}{3} + 2An_1\sqrt{\frac{T}{3}} > -A^2T - 2An_1\sqrt{\frac{T}{3}} \;\rightarrow A\sqrt{\frac{T}{3}} > -n_1 \;,\; \varepsilon_{s_1}^{0.5} > -n_1$$

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_3) : \frac{A^2T}{3} + 2An_1\sqrt{\frac{T}{3}} > \frac{A^2T}{12} + An_1\sqrt{T}\left(\frac{2}{\sqrt{3}} - \frac{\sqrt{3}}{2}\right) + \frac{An_2\sqrt{T}}{2} \;\rightarrow \sqrt{3}\left(\frac{A}{2}\sqrt{\frac{T}{3}} + n_1\right) > n_2 \qquad (16)$$

The implications of the conditions on the first and second lines of (16) are shown in Fig. 6

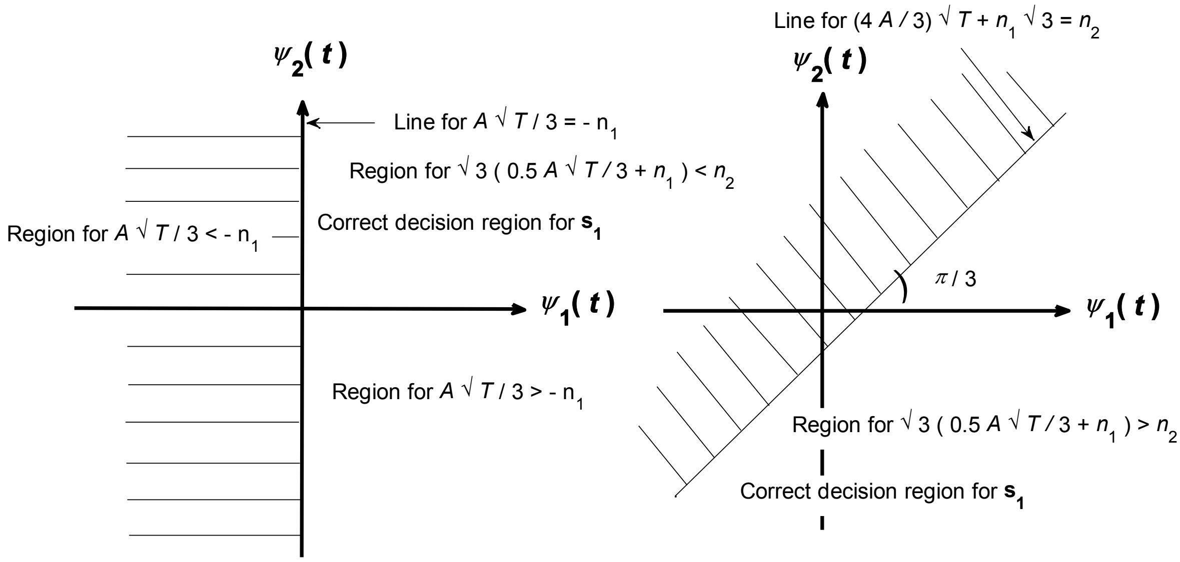


Fig. 6 Regions defined by the conditions in (16) in Q1.

Taking into account the regions defined in Fig. 6 and benefitting from (6.28) and (6.29) of Notes on Dimensionality of Signals_Sept 2012_HTE and (8) – (10) of SampleProblems of Proakis 2002_Nov 2012, we arrive at the probability of correct decision and the probability of error for $s_1(t)$ being sent from the transmitter as shown below

$$P_c(\mathbf{s}_1) = \frac{1}{(\pi N_0)^{0.5}} \int_{-A\sqrt{\frac{T}{3}}}^{\infty} \exp\left(-\frac{n_1^2}{N_0}\right) dn_1 \frac{1}{(\pi N_0)^{0.5}} \int_{-\infty}^{\sqrt{3}\left(\frac{A}{2}\sqrt{\frac{T}{3}}+n_1\right)} \exp\left(-\frac{n_2^2}{N_0}\right) dn_2 \;,\; P_e(\mathbf{s}_1) = 1 - P_c(\mathbf{s}_1) \qquad (17)$$

2. (30 Points) Answer the following questions as **True** or **False.** For the **False** ones give the correct answer or the reason. For the **True** ones, justify your answer

a) In CPM, the number of states are determined together by the modulation index and the M ary value : True, the exact relation is $N_{\Phi_k} = pM^{L-1}$, where is the prime in the rational modulation index, M is the mary value (number of levels in our signal), L is the symbol interval, that the shaping waveform extends over.

b) In CPM in the time duration between the beginning of a symbol and its end, the phase changes continuously : True but such phase change is smooth so that a narrower frequency spectrum than a conventional PSK is generated.

c) CPM is preferred over PSK, since CPM requires narrower bandwidth : True, this is exactly the reason why CPM is preferred over PSK.

d) On the basis of equal energy, and same SNR, PSK has the lowest probability of error, ASK is the next and QAM has the worst probability of error performance : False, for M values equal and greater than eight, the ordering from the lowest to the highest probability of error is QAM, PSK, ASK.

e) Optimum detector operates on the principle of choosing the signal vector nearest to the received signal vector : True, this is why we evaluate the correlation metrics.

f) MF and Matched Filter give the same output : True, since MF is just an abbreviation for the wording Matched Filter.