Çankaya University – ECE Department – ECE 632 (MT)

Student Name : Date : 17.11.2015
Student Number : Open Source Exam

Questions

1. (70 Points) The related constellation is given in Fig. 1.1.

   a. Identify the type of modulation and dimensionality in this constellation. Write and plot the mathematical expression for the basis functions, $\psi_1(t),\ \psi_2(t)$, write for the signal vectors $\mathbf{s}_1 \cdots \mathbf{s}_8$, write and plot the corresponding signal waveforms of $s_1(t) \cdots s_8(t)$. Find the distance between signal vector ends. Determine the total energy used if all signals are sent from the transmitter with equal probability.
   b. Find the probability of error and decision regions via the evaluations of correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ for $\mathbf{s}_1$ and $\mathbf{s}_2$. Comment on how the probability error of the rest of the signals will be related to that of $\mathbf{s}_1$ and $\mathbf{s}_2$.

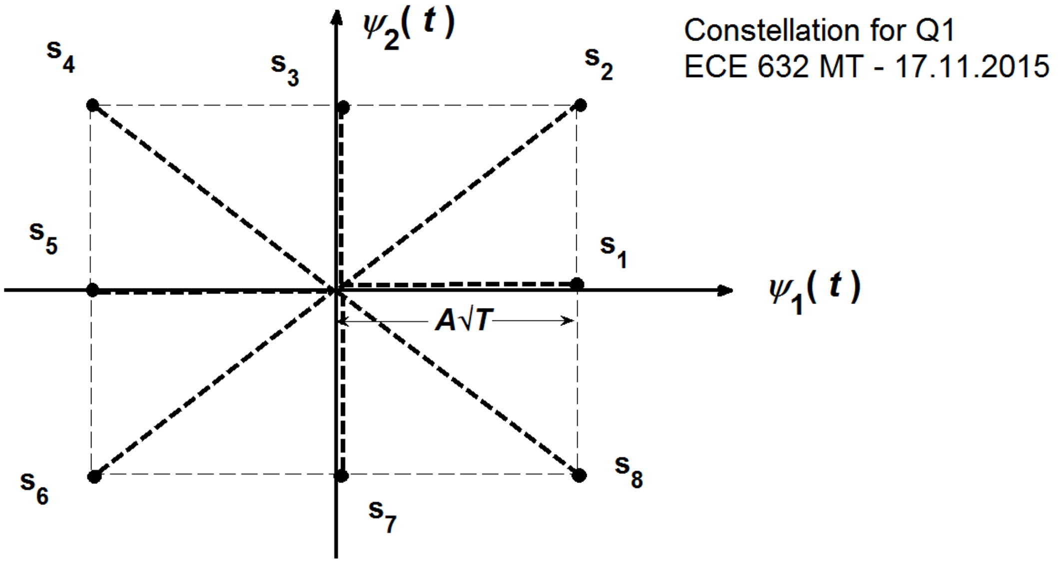

Fig. 1.1 Constellation A.

Solution : a. Looking at the constellation in Fig. 1.1, we see that the signal vector lengths, thus the related energies are unequal and the signal space in this constellation contains two dimensions, thus constellation A is QAM.

By adapting the following orthonormalized basis functions,

$$\psi_1(t) = \begin{cases} \sqrt{2/T} & 0 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} \sqrt{2/T} & T/2 \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

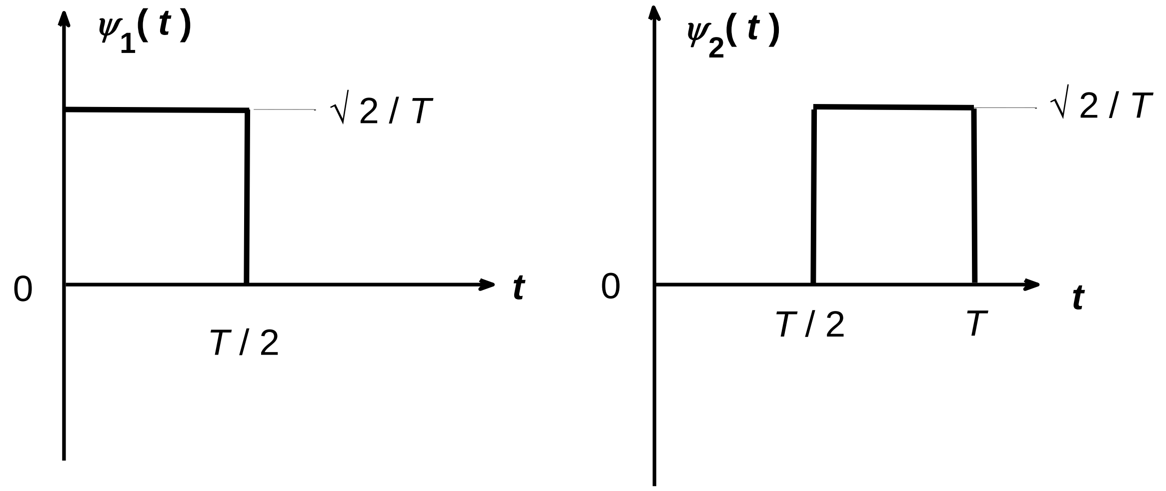

Fig. 1.2 The orthonormalized basis functions for Q1.

By using (1), Figs. 1.1 and 1.2 we obtain the followings for the time waveforms of $s_1(t)\cdots s_8(t)$

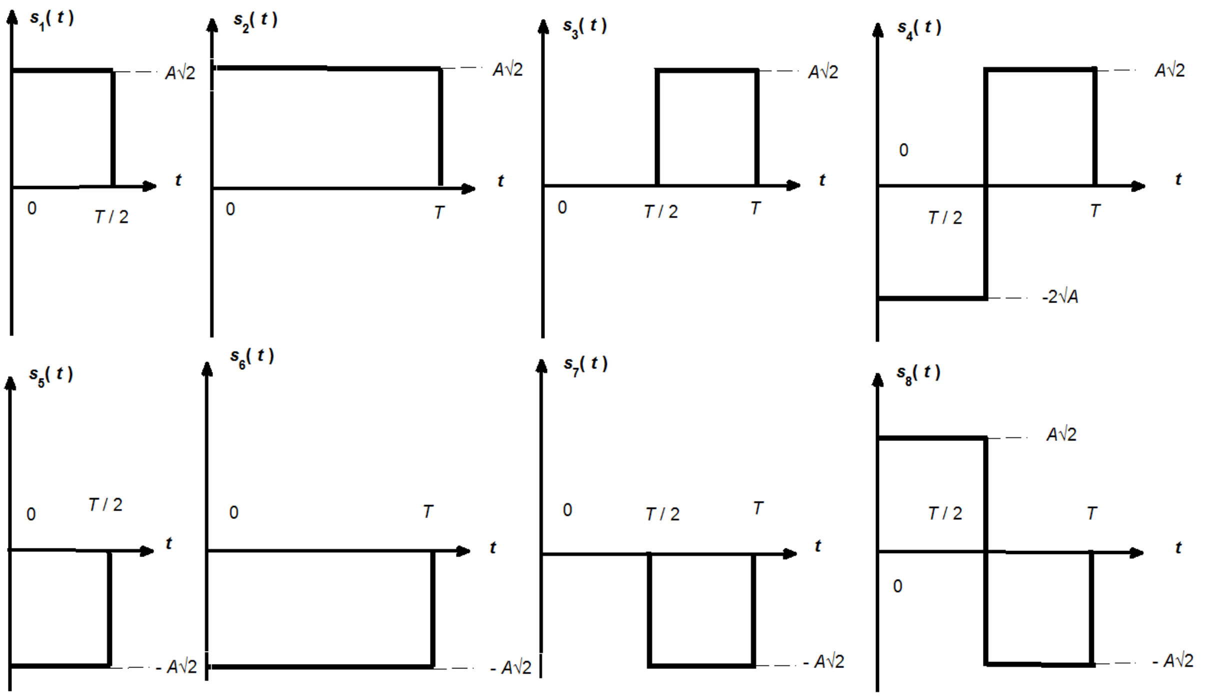

Fig. 1.3 Time waveforms of $s_1(t)\cdots s_8(t)$ for Q1.

The related time waveform expressions for $s_1(t)\cdots s_8(t)$ are given in (1.2).

$$s_1(t)=\begin{cases}A\sqrt{2} & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases},\ s_1(t)=A\sqrt{T}\psi_1(t),\ \mathbf{s_1}=[s_{11},\ s_{12}]=\left[A\sqrt{T},\ 0\right],\ \varepsilon_{s_1^a}=\|\mathbf{s_1}\|^2=A^2T$$

$$s_2(t)=\begin{cases}A\sqrt{2} & 0\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_2(t)=A\sqrt{T}\psi_1(t)+A\sqrt{T}\psi_2(t),\ \mathbf{s_2}=[s_{21},\ s_{22}]=\left[A\sqrt{T},\ A\sqrt{T}\right],\ \varepsilon_{s_2^a}=\|\mathbf{s_2^a}\|^2=2A^2T$$

$$s_3(t)=\begin{cases}A\sqrt{2} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_3(t)=A\sqrt{T}\psi_2(t),\ \mathbf{s_3}=[s_{31},\ s_{32}]=\left[0,\ A\sqrt{T}\right],\ \varepsilon_{s_3^a}=A^2T$$

$$s_4(t)=\begin{cases}-A\sqrt{2} & 0\le t\le T/2\\ A\sqrt{2} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_4(t)=-A\sqrt{T}\psi_1(t)+A\sqrt{T}\psi_2(t),\ \mathbf{s_4}=[s_{41},\ s_{42}]=\left[-A\sqrt{T},\ A\sqrt{T}\right],\ \varepsilon_{s_4^a}=2A^2T$$

$$s_5(t)=\begin{cases}-A\sqrt{2} & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases},\ s_5(t)=-A\sqrt{T}\psi_1(t),\ \mathbf{s_5}=[s_{51},\ s_{52}]=\left[-A\sqrt{T},\ 0\right],\ \varepsilon_{s_5^a}=A^2T$$

$$s_6^a(t)=\begin{cases}-A\sqrt{2} & 0\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_6(t)=-A\sqrt{T}\psi_1(t)-A\sqrt{T}\psi_2(t),\ \mathbf{s_6}=[s_{61},\ s_{62}]=\left[-A\sqrt{T},\ -A\sqrt{T}\right],\ \varepsilon_{s_6^a}=2A^2T$$

$$s_7(t)=\begin{cases}-A\sqrt{2} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_7(t)=-A\sqrt{2}\psi_2(t),\ \mathbf{s_7}=[s_{71},\ s_{72}]=\left[0,\ -A\sqrt{2}\right],\ \varepsilon_{s_7^a}=A^2T$$

$$s_8(t)=\begin{cases}A\sqrt{2} & 0\le t\le T/2\\ -A\sqrt{2} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_8(t)=A\sqrt{2}\psi_1(t)-A\sqrt{2}\psi_2(t),\ \mathbf{s_8}=[s_{81},\ s_{82}]=\left[A\sqrt{2},\ -A\sqrt{2}\right],\ \varepsilon_{s_8^a}=2A^2T \quad (1.2)$$

The total energy used if all signals are transmitted with equal probability, $E_T=4\times A^2T+4\times 2A^2T=12A^2T$

The distances between signal vector ends are given in (1.3)

$$\begin{aligned}&d_{12}=d_{23}=d_{34}=d_{45}=d_{56}=d_{67}=A\sqrt{T}\\&d_{13}=d_{35}=d_{57}=d_{71}=A\sqrt{2T}\\&d_{24}=d_{46}=d_{68}=d_{82}=d_{15}=d_{37}=2A\sqrt{T}\\&d_{26}=d_{48}=2A\sqrt{2T}\\&d_{14}=d_{16}=d_{38}=A\sqrt{5T}\end{aligned} \quad (1.3)$$

b. Looking at constellation A, we see that there are two types of decision boundaries. They can be described by examining the cases of $\mathbf{s_1}$ and $\mathbf{s_2}$. The received vector, $\mathbf{r}$ that we supply to the detector for the two cases will become

$$\mathbf{r_1}=\begin{bmatrix}r_{11}\\ r_{12}\end{bmatrix}=\begin{bmatrix}A\sqrt{T}+n_1\\ n_2\end{bmatrix}\quad,\quad \mathbf{r_2}=\begin{bmatrix}r_{21}\\ r_{22}\end{bmatrix}=\begin{bmatrix}A\sqrt{T}+n_1\\ A\sqrt{T}+n_2\end{bmatrix} \quad (1.4)$$

Based on ECE 588MT-06042015_Solutions, it is sufficient to evaluate correlation metrics $C_1(\mathbf{r_1},\mathbf{s_m})$ at $m=1,\ 2,\ 8,\ 5$ for $\mathbf{s_1}$, $C_2(\mathbf{r_2},\mathbf{s_m})$ at $m=1,\ 2,\ 3,\ 6$ and for $\mathbf{s_2}$, the others will be covered in these analyses.

$$m=1\ ,\ C_1(\mathbf{r_1},\mathbf{s_1})=2\,\mathbf{s_1}\cdot\mathbf{r_1}-\|\mathbf{s_1}\|^2=2\left[A\sqrt{T},\ 0\right]\begin{bmatrix}A\sqrt{T}+n_1\\ n_2\end{bmatrix}-A^2T=A^2T+2An_1\sqrt{T}$$

$$m=2\ ,\ C_1(\mathbf{r_1},\mathbf{s_2})=2\,\mathbf{s_2}\cdot\mathbf{r_1}-\|\mathbf{s_2}\|^2=2\left[A\sqrt{T},\ A\sqrt{T}\right]\begin{bmatrix}A\sqrt{T}+n_1\\ n_2\end{bmatrix}-2A^2T=2An_1\sqrt{T}+2An_2\sqrt{T}$$

$$m=5\ ,\ C_1(\mathbf{r_1},\mathbf{s_5})=2\,\mathbf{s_5}\cdot\mathbf{r_1}-\|\mathbf{s_5}\|^2=2\left[-A\sqrt{T},\ 0\right]\begin{bmatrix}A\sqrt{T}+n_1\\ n_2\end{bmatrix}-A^2T=-3A^2T-2An_1\sqrt{T}$$

$$m=8\ ,\ C_1(\mathbf{r_1},\mathbf{s_8})=2\,\mathbf{s_8}\cdot\mathbf{r_1}-\|\mathbf{s_8}\|^2=2\left[A\sqrt{T},\ -A\sqrt{T}\right]\begin{bmatrix}A\sqrt{T}+n_1\\ n_2\end{bmatrix}-2A^2T=2An_1\sqrt{T}-2An_2\sqrt{T}$$

$$m=1\ ,\ C_2(\mathbf{r_2},\mathbf{s_1})=2\,\mathbf{s_1}\cdot\mathbf{r_2}-\|\mathbf{s_1}\|^2=2\left[A\sqrt{T},\ 0\right]\begin{bmatrix}A\sqrt{T}+n_1\\ A\sqrt{T}+n_2\end{bmatrix}-A^2T=A^2T+2An_1\sqrt{T}$$

$$m=2\ ,\ C_2(\mathbf{r_2},\mathbf{s_2})=2\,\mathbf{s_2}\cdot\mathbf{r_2}-\|\mathbf{s_2}\|^2=2\left[A\sqrt{T},\ A\sqrt{T}\right]\begin{bmatrix}A\sqrt{T}+n_1\\ A\sqrt{T}+n_2\end{bmatrix}-2A^2T=2A^2T+2An_1\sqrt{T}+2An_2\sqrt{T}$$

$$m=3\ ,\ C(\mathbf{r_2},\mathbf{s_3})=2\,\mathbf{s_3}\cdot\mathbf{r_2}-\|\mathbf{s_3}\|^2=2\left[0,\ A\sqrt{T}\right]\begin{bmatrix}A\sqrt{T}+n_1\\ A\sqrt{T}+n_2\end{bmatrix}-A^2T=A^2T+2An_2\sqrt{T}$$

$$m=6\ ,\ C(\mathbf{r_2},\mathbf{s_6})=2\,\mathbf{s_6}\cdot\mathbf{r_2}-\|\mathbf{s_6}\|^2=2\left[-A\sqrt{T},\ -A\sqrt{T}\right]\begin{bmatrix}A\sqrt{T}+n_1\\ A\sqrt{T}+n_2\end{bmatrix}-2A^2T=-6A^2T-2An_1\sqrt{T}-2An_2\sqrt{T}\quad(1.5)$$

To get the correct decision regions we must have

$$\text{For } \mathbf{s_1}\ :C_1(\mathbf{r_1},\mathbf{s_1})>C_1(\mathbf{r_1},\mathbf{s_2}),\ C_1(\mathbf{r_1},\mathbf{s_1})>C_1(\mathbf{r_1},\mathbf{s_8}),\ C_1(\mathbf{r_1},\mathbf{s_1})>C_1(\mathbf{r_1},\mathbf{s_5})$$
$$\text{For } \mathbf{s_2}\ :C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_1},\mathbf{s_1}),\ C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_2},\mathbf{s_3}),\ C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_2},\mathbf{s_6})\qquad(1.6)$$

Hence, we get the following inequalities

$$C_1(\mathbf{r_1},\mathbf{s_1})>C_1(\mathbf{r_1},\mathbf{s_2})\ :A^2T+2An_1\sqrt{T}>2An_1\sqrt{T}+2An_2\sqrt{T}\ \rightarrow \underbrace{A\sqrt{T}}_{r_{11}-n_1}>2\underbrace{n_2}_{r_{12}}\ \rightarrow 0.5r_{11}-0.5n_1>r_{12}$$
$$C_1(\mathbf{r_1},\mathbf{s_1})>C_1(\mathbf{r_1},\mathbf{s_5})\ :A^2T+2An_1\sqrt{T}>-3A^2T-2An_1\sqrt{T}\ \rightarrow\ r_{11}>0$$
$$C_1(\mathbf{r_1},\mathbf{s_1})>C_1(\mathbf{r_1},\mathbf{s_8})\ :A^2T+2An_1\sqrt{T}>2An_1\sqrt{T}-2An_2\sqrt{T}\ \rightarrow\ -0.5r_{11}+0.5n_1<r_{12}$$

$$C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_2},\mathbf{s_1})\ :2A^2T+2An_1\sqrt{T}+2An_2\sqrt{T}>A^2T+2An_1\sqrt{T}\ \rightarrow A\sqrt{T}+2\overbrace{n_2}^{r_{22}-A\sqrt{T}}>0\ \rightarrow r_{22}>0.5A\sqrt{T}$$
$$C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_2},\mathbf{s_3})\ :2A^2T+2An_1\sqrt{T}+2An_2\sqrt{T}>A^2T+2An_2\sqrt{T}\ \rightarrow\ r_{21}>0.5A\sqrt{T}$$
$$C_2(\mathbf{r_2},\mathbf{s_2})>C_2(\mathbf{r_2},\mathbf{s_6})\ :2A^2T+2An_1\sqrt{T}+2An_2\sqrt{T}>6A^2T-2An_1\sqrt{T}-2An_2\sqrt{T}\ \rightarrow\ r_{22}>-r_{21}\qquad(1.7)$$

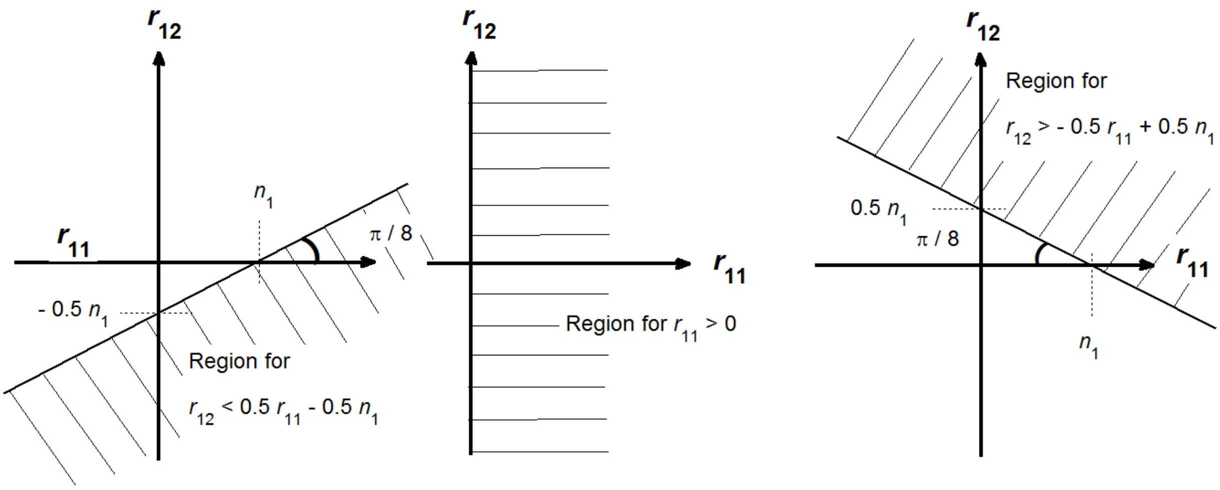

Fig. 1. 4 Regions for conditions in (1.7) for $\mathbf{s}_1$.

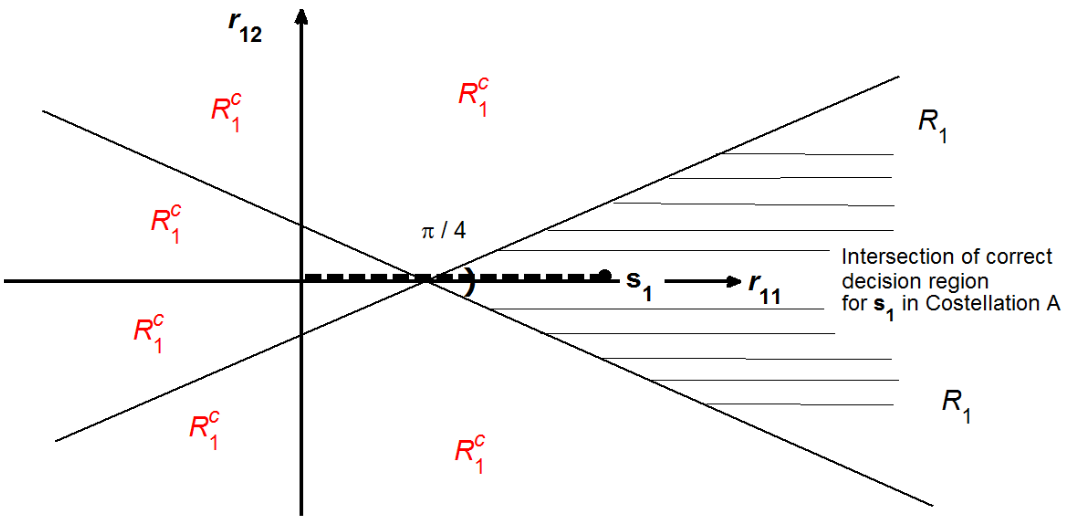

Fig. 1. 5 Intersection of correct decision region for conditions in (1.7). for $\mathbf{s}_1$.

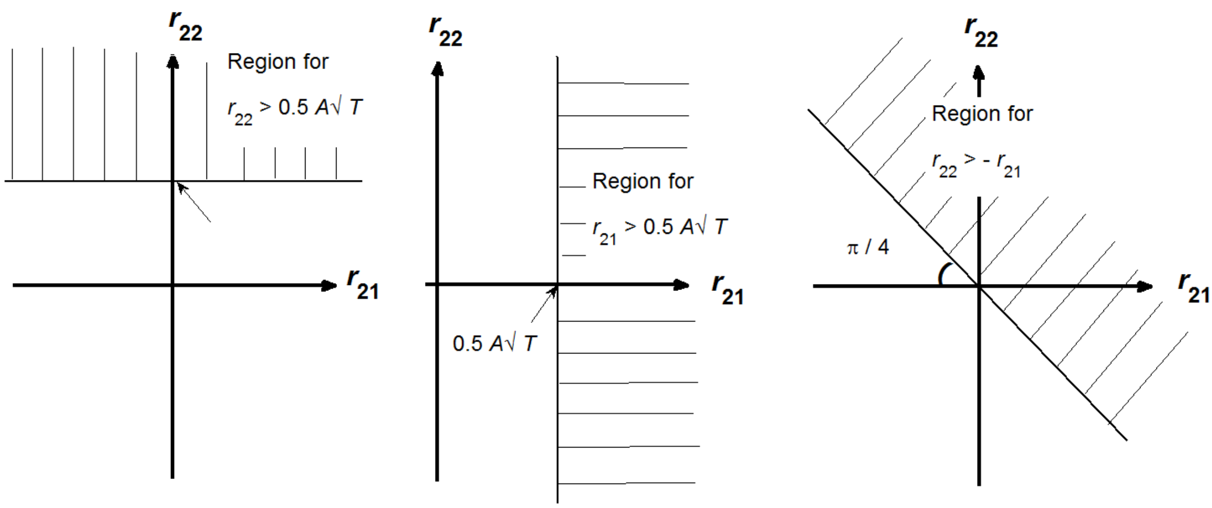

Fig. 1. 6 Regions for conditions in (1.7) for $\mathbf{s}_2$.

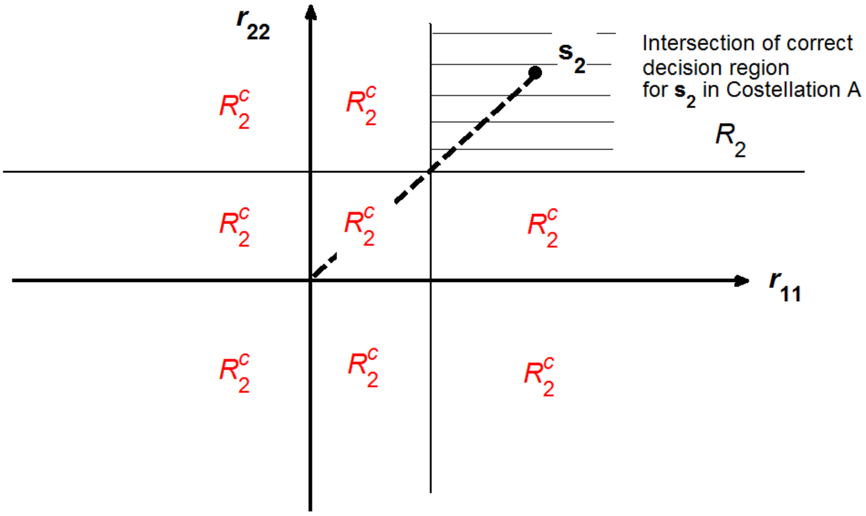

Fig. 1. 7 Intersection of correct decision region for conditions in (1.7). for $\mathbf{s}_2$.

Signal vectors, $\mathbf{s}_1$, $\mathbf{s}_3$, $\mathbf{s}_5$ and $\mathbf{s}_7$ will have identical correct decision regions, thus the probability of error for $\mathbf{s}_1$, $\mathbf{s}_3$, $\mathbf{s}_5$ and $\mathbf{s}_7$ will be the same. On the other hand, signal vectors, $\mathbf{s}_2$, $\mathbf{s}_4$, $\mathbf{s}_6$ and $\mathbf{s}_8$ will have identical correct decision regions, thus the probability of error for $\mathbf{s}_2$, $\mathbf{s}_4$, $\mathbf{s}_6$ and $\mathbf{s}_8$ will be the same.

2. (30 Points) Answer the following questions as **True** or **False.** For the **False** ones give the correct answer or the reason. For the **True** ones, justify your answer

a) PSK arranges the distribution of signal vectors over a signal space diagram of 3 dimensions : False, PSK is 2 dimensional.

b) ASK has the worst probability of error performance : Correct, according to results of Exercise 6.2 of lecture notes entitled, " Notes on Dimensionality of Signals_Sept 2012_HTE".

c) QAM is used in radio links : True, since QAM uses the signal space diagram more efficiently than ASK and PSK.

d) For multidimensional signals, the complexity of the demodulator increases : True, since the number of branches in the correlator or matched filter increases.

e) The construction of the demodulator depends on the dimensionality of the transmitted signals and is not related to modulation type : True, according to Figs. 5.2 and 5.3 lecture notes entitled, " Notes on Dimensionality of Signals_Sept 2012_HTE". But in the evaluation of correlation metrics values, knowledge of modulation type is essential.

f) Detection means the evaluation of $C\left(\mathbf{r}, \mathbf{s}_m\right)$ correlation metrics values and deciding on the minimum : First part is true, according to (6.9) of lecture notes entitled, " Notes on Dimensionality of Signals_Sept 2012_HTE", but as seen from there, max is chosen, not the minimum.